MATH1961 Mathematics 1A (Advanced)

UNIVERSITY OF SYDNEY · SCHOOL OF MATHEMATICS & STATISTICS

EXAM REVISION
Sem 1 2026 · **SIDE 1 OF 2**
Rigour & calculus · prove it

SIDE 1/2 RIGOUR · Proof toolkit & induction · ε–δ limits · Completeness & sup · Continuity · IVT/EVT · Rolle→MVT→Taylor · Riemann & FTC

REVISION SHEET · ALL TOPICS

Compiled by **AskSia** · mapped to the MATH1961 syllabus · asksia.ai/cheatsheet/usyd-math1961

0 · Exam Blueprint

READ FIRST

★ PROVE, don't just compute. This is the *Advanced* unit (≈4 h/wk vs 3): nearly every definition is stated rigorously and almost every theorem is **proved**. The 60% exam rewards a correct ε–δ argument, a clean induction, a fully-justified IVT/MVT, and linear-algebra *reasoning* — not the final number alone.

Assessment: exam 60% · mid-sem quiz 13% · 10 online quizzes 10% · 2 assignments 15% · tutorials 2%. Both streams (Calculus + Linear Algebra) on one paper.

Where marks are won: ε–δ limit proofs · sup/inf completeness · induction (weak & strong) · the Rolle→MVT→Taylor→FTC chain · rank–nullity & eigenvalue reasoning. These are exactly what mainstream 1061 can't do.

Two streams, one paper: Calculus (Cirstea, Daners notes) runs complex numbers → limits → continuity → differentiation → Taylor → integration; Linear Algebra (Brownlowe) runs vectors → systems → matrices → subspaces → transformations → eigenvalues. Reproduce named proofs verbatim where you can — they recur.

SIA → *Style is graded (LO2): write proofs as full English sentences mixing words + symbols. Pure symbol-soup OR pure prose both lose marks even with the right idea. State the definition first, name the technique, then argue.*

1 · Proof Toolkit

NAME THE METHOD

Quantifiers & negation — get these exact or lose partial credit. $\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$; $\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$. Prove $\exists x{:}P(x)$ by *exhibiting one witness*; prove $\forall x{:}P(x)$ by a *generalised* argument on arbitrary x; **disprove ∀** by a single counterexample.

THREE GENERALISED PROOFS

- **Direct** — assume P, deduce Q. e.g. $m=j^2$, $n=k^2 \Rightarrow mn=(jk)^2$.
- **Contrapositive** — to prove $P\Rightarrow Q$, prove $\neg Q\Rightarrow\neg P$. e.g. n^2 even ⇒ n even (assume $n=2k+1 \Rightarrow n^2$ odd).
- **Contradiction** — assume $P \wedge \neg Q$, derive absurdity. Classic: $\sqrt{2}$ irrational.
- **Cases** — for (P or Q)⇒R, prove each branch.

⚠ A counterexample *is* a complete proof of a "false/disprove" claim.

Notation marks: ∈ "element of"; := "defined to be" (vs = a deduced equality); distinguish ⇒, ⇐, ⇔. Number systems $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$. Set-builder $\{x\in\mathbb{Z} : 1\le x\le 5\}$. Sloppy quantifier order or arrows lose marks even when the idea is right.

1b · Induction Template

EXAMINED EVERY SEM

Weak induction. P(n) for all $n\in\mathbb{N}$ if: **(Basis)** P(0) (or P(1)) holds; **(Step)** assume P(k) [the *inductive hypothesis*], prove P(k+1).

> **SKELETON**
> 1. State P(n) precisely.
> 2. Basis: verify P(0).
> 3. Step: "Assume P(k). Then …" ⇒ P(k+1).
> 4. By induction P(n) ∀n. ∎

Canonical: $3 \mid 4^n-1$, via $4^{k+1}-1 = 4(4^k-1)+3$.

Strong (complete) induction. Basis P(0),P(1); step: *if P(i) holds for all 0≤i≤k then P(k+1).* Use strong induction when P(k+1) needs several earlier cases — e.g. order-≥2 recurrences (Fibonacci-type $f(m+2)=f(m+1)+f(m)$).

2 · The ε–δ Limit

CENTRAL DEFINITION

f defined on an open interval around a (not necessarily at a).

> **LIM(X→A) F(X) = L**
> $\forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0 :$
> $0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$

Variants (all examinable):

> $\lim_{x\to a} f = +\infty : \forall M\ \exists\delta>0,\ 0<|x-a|<\delta \Rightarrow f(x)>M$
> $\lim_{x\to\infty} f = L : \forall\varepsilon>0\ \exists N,\ x>N \Rightarrow |f(x)-L|<\varepsilon$

plus one-sided ($x\to a^+, a^-$) and ±∞ point/value combinations. The quantifier ORDER is the marked part.

2b · ε–δ Proof Skeleton

PROVE A LIMIT

> **METHOD**
> 1. Fix ε>0 (arbitrary).
> 2. Bound |f(x)-L| ≤ (expr in |x-a|).
> 3. Choose δ so that bound < ε.
> 4. Verify 0<|x-a|<δ ⇒ |f(x)-L|<ε. ∎

Worked: $\lim_{x\to a}\sqrt{x}=\sqrt{a}$ (a>0). $|\sqrt{x}-\sqrt{a}| = |x-a|/(\sqrt{x}+\sqrt{a}) \le |x-a|/\sqrt{a}$. So take $\delta=\varepsilon\sqrt{a}$: then $|x-a|<\delta \Rightarrow |\sqrt{x}-\sqrt{a}| \le \delta/\sqrt{a}=\varepsilon$. ∎

Squeeze law: $f \le h \le g$ near a and $\lim f = \lim g = L \Rightarrow \lim h = L$. Kills $x\cdot\cos(1/x)\to 0$. **Fundamental limit:** $\lim_{x\to 0} \sin x / x = 1$.

DNE by two sequences ⚠ : if $x_n\to a$, $\tilde{x}_n\to a$ give $\lim f(x_n) \ne \lim f(\tilde{x}_n)$, the limit does not exist. You must *exhibit both sequences* — don't just assert oscillation.

3 · Continuity

LIMIT = VALUE

f is **continuous at a** if $\lim_{x\to a} f(x)$ exists *and* equals f(a). One-sided: right-cts at a if $\lim_{x\to a^+} f = f(a)$. Continuous on A = cts at each point.

⚠ A limit can exist yet f be discontinuous because f(a) ≠ lim. Piecewise "find a,b so f continuous": match one-sided limits to the value.

Substitution (composition): f cts at m and $\lim_{x\to a} g = m \Rightarrow \lim_{x\to a} f(g(x)) = f(m)$. Proof chains two ε–δ statements.

Worked (DNE): $\lim_{x\to 0^+} \cos(1/\sqrt{x})$ fails to exist. Take $x_n = 1/(4n^2\pi^2) \to 0$ with cos→1, and $\tilde{x}_n = 1/(\pi/2+2n\pi)^2 \to 0$ with cos→0. Two limits differ ⇒ no limit. ∎

3b · Limit Laws

USE AFTER PROVING ONCE

Once a few limits are proved from the definition, combine by the laws: $\lim(f\pm g)=\lim f \pm \lim g$; $\lim(fg)=\lim f\cdot\lim g$; $\lim(f/g)=\lim f / \lim g$ (denominator limit ≠ 0). Polynomials & rationals are continuous on their domain, so limits = substitution there.

⚠ Laws presuppose each piece's limit *exists* — never split a limit you haven't shown exists (0·∞, ∞−∞ traps).

Asymptotes from limits: $\lim_{x\to a} f = \pm\infty \Rightarrow$ vertical asymptote x=a; $\lim_{x\to\pm\infty} f = L \Rightarrow$ horizontal asymptote y=L. e.g. arctan x has horizontals $y=\pm\pi/2$; 1/x has both.

Worked squeeze: show $\lim_{x\to 0} x^2\sin(1/x) = 0$. Since $-x^2 \le x^2\sin(1/x) \le x^2$ and both bounds → 0, the squeeze law gives 0. ∎ (note sin(1/x) alone has no limit at 0 — the x^2 factor is what forces it.) Likewise $\lim_{x\to\infty} \cos x/x = 0$ by squeeze between ±1/x.

The sequential criterion (DNE via two sequences) is also the bridge to limits of sequences — the same ε–N machinery underlies both.

4 · Completeness · sup/inf

THE CORE OF ℝ

$A\subseteq\mathbb{R}$ nonempty is **bounded above** if $\exists b: a\le b\ \forall a\in A$ (b an *upper bound*); dually bounded below.

> **C = SUP A (LEAST UPPER BOUND)**
> (1) $a \le c\ \forall a\in A$ [c is an upper bound]
> (2) ∀ upper bound b : $c \le b$ [least one]

β = inf A = greatest lower bound (dual). **LUB Axiom** ⚠ (defining property of ℝ): every nonempty $A\subseteq\mathbb{R}$ bounded above has a supremum in ℝ (dual GLB axiom for inf).

sup A need not lie in A: $A=\{1-1/n\}$ has sup $A=1\notin A$. If sup $A\in A$ then sup A = max A.

4b · How to PROVE c = sup A

BOTH CLAUSES

> **TWO-CLAUSE RECIPE**
> (1) Show $a \le c$ for every $a\in A$.
> (2) Show no smaller bound: $\forall\varepsilon>0\ \exists a\in A$ with $c-\varepsilon < a \le c$.

Clause (2) says "c−ε is *not* an upper bound for any ε>0" — the standard way to pin the *least* bound. ⚠ Verifying only clause (1) is the most common lost-mark.

Approximating sequence: if c=sup A then taking ε=1/n gives $a_n\in A$ with $c-1/n < a_n \le c$, so $a_n\to c$. A monotone increasing sequence in A also reaches sup A.

5 · IVT & EVT

PROVED VIA SUP

Intermediate Value Thm ⚠ . $f:[a,b]\to\mathbb{R}$ continuous ⇒ for every ℓ strictly between f(a),f(b), $\exists x_1\in(a,b): f(x_1)=\ell$.

Proof idea: WLOG f(a)<ℓ<f(b). Set $A=\{x\in(a,b): f<\ell \text{ on } [a,x)\}$, let $x_1=\sup A$ (completeness!). Continuity squeezes $f(x_1)=\ell$ from both sides. ∎

Use: "show f has a root" → sign change at two points (or limits at ±∞), then IVT.

Extreme Value Thm ⚠ . f continuous on *closed bounded* [a,b] ⇒ ∃ x_m, x_M with $f(x_m) \le f \le f(x_M)$ (min & max attained), so Range = $[f(x_m),f(x_M)]$.

⚠ EVT needs **closed + bounded + continuous**; drop any (open interval, jump) and max/min can fail. **Surjectivity:** cts f with $\lim_{-\infty}=-\infty$, $\lim_{+\infty}=+\infty$ ⇒ IVT gives every value.

5b · Worked · sup proof

BOTH CLAUSES

Claim: $A = \{1-1/n : n\ge 1\}$ has sup A = 1.

(1) Upper bound: $1-1/n < 1$ for all n≥1, so 1 is an upper bound.

(2) Least: take any ε>0. Pick n with 1/n < ε (Archimedean). Then $1-1/n > 1-\varepsilon$, so 1−ε is *not* an upper bound. Hence no number < 1 bounds A. ⇒ sup A = 1 (and $1\notin A$, so A has no maximum). ∎ Note inf A = 0 (attained at n=1, so inf A = min A = 0).

⚠ Conventions: A unbounded above ⇒ sup A = +∞; A = ∅ ⇒ sup A = −∞. These edge cases appear in "state the sup" questions. The EVT also proves: a continuous function on a closed bounded interval is bounded — boundedness comes first, then the bounds are attained.

5c · Worked · IVT root

EXAM REFLEX

Show $x^5+x-1=0$ has a real root. Let $f(x)=x^5+x-1$, continuous. f(0)=−1<0, f(1)=1>0; by IVT ∃ c∈(0,1) with f(c)=0. $f'(x)=5x^4+1>0$ ⇒ f strictly increasing ⇒ exactly one root. ∎ This "IVT for existence + monotonicity for uniqueness" pairing is the standard root-counting answer.

6 · Differentiation

RIGOROUS DEF

> **F'(X₀)**
> $f'(x_0) = \lim_{x\to x_0} [f(x)-f(x_0)]/(x-x_0) \in \mathbb{R}$

Differentiable ⇒ continuous (not conversely). **Carathéodory form:** ∃ m continuous at x_0 with $f(x)=f(x_0)+m(x)(x-x_0)$, and $f'(x_0)=m(x_0)$ — makes product/chain-rule proofs clean.

> **RULES**
> $(f+g)'=f'+g'$ · $(fg)'=f'g+fg'$
> $(f/g)'=(f'g-fg')/g^2$ · $(f\circ g)'=f'(g)\cdot g'$

Critical point: $f'(x_0)=0$ or undefined. Extrema are critical, but not every critical point is an extremum (x^3 at 0). 2nd-deriv test: f''>0 min, <0 max, =0 inconclusive.

7 · The MVT Chain

ROLLE→… →L'HÔPITAL

Rolle ⚠ . f cts [a,b], diff (a,b), f(a)=f(b) ⇒ ∃c∈(a,b): f'(c)=0. *Proof:* EVT gives extrema; an interior extremum forces f'(c)=0 (else f const).

Cauchy MVT ⚠ . ∃c: $[f(b)-f(a)]g'(c) = [g(b)-g(a)]f'(c)$. *Proof:* apply Rolle to $F(x)=[f(b)-f(a)][g(x)-g(a)]-[g(b)-g(a)][f(x)-f(a)]$ (F(a)=F(b)=0).

> **MVT (TAKE G(X)=X)**
> $\exists c\in(a,b):\ f'(c) = [f(b)-f(a)]/(b-a)$

Corollaries: f'≡0 ⇒ f const; f'>0 ⇒ strictly ↑; f'<0 ⇒ strictly ↓ (proof: MVT on subintervals). "How many roots?" → f' one sign ⇒ injective ⇒ ≤1 root, + IVT for existence.

L'Hôpital ⚠ (from Cauchy MVT). 0/0 form, g'≠0, lim f'/g'=L ⇒ lim f/g = L. *Proof:* set f(a)=g(a)=0, apply Cauchy MVT on [a,x].

7b · Worked · MVT use

INEQUALITY

Prove |sin a − sin b| ≤ |a−b|. Apply MVT to f(x)=sin x on [b,a]: sin a − sin b = cos(c)·(a−b) for some c. Since |cos c| ≤ 1, |sin a − sin b| ≤ |a−b|. ∎

Monotone count: "how many roots has 3x − sin x?" f'(x)=3−cos x ≥ 2 > 0 ⇒ strictly increasing ⇒ at most one root; f(0)=0 gives exactly one.

7c · Implicit Diff & L'Hôpital

TECHNIQUE

Implicit: differentiate the relation in x treating y=y(x), then solve dy/dx. e.g. $x^2y^2 + x\sin y = 4 \Rightarrow dy/dx = -(2xy^2 + \sin y)/(2x^2y + x\cos y)$.

L'Hôpital worked: $\lim_{x\to 0}(1-\cos x)/x^2 = \lim \sin x/2x = \lim \cos x/2 = ½$ (two applications, each a 0/0 form). Check the indeterminate form before each step.

7d · Diff ⇒ Cts, not conversely

CLASSIC TRAP

f(x)=|x| is continuous at 0 but **not** differentiable: the left difference quotient → −1, the right → +1, so f'(0) does not exist. Differentiable ⇒ continuous; continuous ⇏ differentiable.

2nd-deriv test inconclusive case: $f(x)=x^4$ at 0 has f'=f''=0 yet a min — fall back to the sign change of f'.

Rolle in one line: any polynomial of degree n has at most n real roots — between consecutive roots f'(c)=0 by Rolle, so the derivative (degree n−1) caps the root count by induction.

The product/chain proofs run cleanest through the Carathéodory form: write each factor as $f(x_0)+m(x)(x-x_0)$ and read off the derivative from $m(x_0)$.

Singular critical points (f′ undefined) count too: |x| at 0 is a minimum with no derivative — always check both stationary and singular points.

8 · Taylor + Lagrange

HIGH-VALUE PROOF

> **NTH TAYLOR POLY AT X₀**
> $T_n(x) = \sum_{k=0}^{n} f^{(k)}(x_0)/k! \cdot (x-x_0)^k$

T_n is the *unique* degree-≤n poly P with $\lim_{x\to x_0} [f-P]/(x-x_0)^n = 0$ — so build it by substitution / multiply / integrate, no repeated differentiation.

> **LAGRANGE REMAINDER** ⚠
> $R_n(x) = f^{(n+1)}(c)/(n+1)! \cdot (x-x_0)^{n+1}$
> for some c between x_0 and x

Proof: Cauchy MVT on $F(t)=\sum_{k=0}^{n} f^{(k)}(t)/k!\cdot(x-t)^k$ and $G(t)=-(x-t)^{n+1}$; the sum telescopes to $F'(t)=f^{(n+1)}(t)/n!\cdot(x-t)^n$. ∎

> **ERROR BOUND (THE EXAM USE)**
> $|R_n(x)| \le \max|f^{(n+1)}| / (n+1)! \cdot |x-x_0|^{n+1}$

e.g. ln(1.1) to $<10^{-4}$: smallest n with $1/[(n+1)10^{n+1}] < 10^{-4} \Rightarrow n=3$.

8b · Standard Series

MEMORISE · ALL AT 0

> $e^x = \sum x^k/k!$ · $1/(1-x) = \sum x^k$ (|x|<1)
> $\sin x = \sum (-1)^k x^{2k+1}/(2k+1)!$
> $\cos x = \sum (-1)^k x^{2k}/(2k)!$
> $\ln(1+x) = \sum (-1)^k x^{k+1}/(k+1)$ (|x|<1)

Parity: sin/arctan (odd) → odd powers; cos (even) → even powers. ⚠ Taylor *poly* always exists; the *series* represents f only if $R_n\to 0$ — must be shown (e^x via Lagrange + $|x|^{n+1}/(n+1)!\to 0$).

8c · Build Without Differentiating

EXAMINED METHODS

- **Substitution:** order-n poly of $f(ax^m) = T_n(ax^m)$ (order mn). e.g. $e^{x^2} = \sum x^{2k}/k!$.
- **Integrate/differentiate:** integrate $1/(1+t^2)=\sum(-1)^k t^{2k} \Rightarrow \arctan x = \sum(-1)^k x^{2k+1}/(2k+1)$.
- **Multiply:** multiply two polys, keep terms up to order n.

Uniqueness (the limit characterisation) guarantees any method gives the same Taylor polynomial. Also $\arctan x = \sum(-1)^k x^{2k+1}/(2k+1)$ on |x|<1.

8d · Convergence

RADIUS

Geometric prototype: $1/(1-x) = \lim(1+x+\dots+x^n)$, remainder $x^{n+1}/(1-x) \to 0$ for |x|<1.

Radius: factorial coefficients (e^x, e^{-x^2}, sin, cos) converge ∀x; geometric-type ($1/(1+x^2)$) only on (−1,1). Representing f always needs $R_n\to 0$, proved case-by-case.

8e · Worked · Estimate e^x

LAGRANGE BOUND

Estimate $e^{0.5}$ with T_3 and bound the error. $T_3(0.5) = 1 + 0.5 + 0.125 + 0.0208\ldots = 1.6458$. Since $f^{(4)}=e^x$ and $e^{0.5}<2$, $|R_3| \le 2/4! \cdot 0.5^4 = 2/24 \cdot 0.0625 \approx 0.0052$. So $e^{0.5} \approx 1.646 \pm 0.006$ (true 1.6487). ∎

Taylor poly of degree 2 at 0 for $f(x)=\sqrt{1+x}$: f(0)=1, f'(0)=½, f''(0)=−¼, so $T_2 = 1 + x/2 - x^2/8$. Limits of indeterminate forms can be read straight off the leading Taylor terms (a fast alternative to L'Hôpital). e.g. $\lim_{x\to 0}(\sin x - x)/x^3$: $\sin x = x - x^3/6 + \dots$, so the quotient → −1/6. Reading the first non-vanishing term beats repeated L'Hôpital.

⚠ A function can be infinitely differentiable yet its Taylor series not represent it away from x_0 — convergence of the series and equality with f are separate facts.

9 · Riemann Integral

DEF + DARBOUX

Partition P: $a=x_0<\dots<x_n=b$, $\Delta x_k=x_k-x_{k-1}$, norm $\|P\|=\max \Delta x_k$. Riemann sum $S = \sum f(x_k^*)\Delta x_k$.

> **INTEGRABLE**
> bounded f is integrable if $\lim_{\|P\|\to 0} S = A$ exists (same A ∀ partitions & samples); $\int_a^b f := A$

Darboux ⚠ : with $U(f,P)=\sum(\sup f)\Delta x_k$, $L(f,P)=\sum(\inf f)\Delta x_k$, f integrable ⟺ ∀ε>0 ∃P: U−L < ε; then $L \le \int \le U$. Continuous ⇒ integrable; monotone ⇒ integrable. ⚠ Bounded ⇏ integrable (Dirichlet $\mathbb{1}_{\mathbb{Q}}$: U=1, L=−1 always). Unbounded ⇒ not integrable.

From definition: for f(x)=x on [a,b], equidistant partition gives $U_n, L_n \to (b^2-a^2)/2$ using $\sum k = n(n+1)/2$, so $\int_a^b x\,dx = (b^2-a^2)/2$.

9b · Integral Properties

F,G INTEGRABLE

> $\int(kf+\ell g) = k\int f + \ell\int g$ (linearity)
> $\int_a^b f = \int_a^c f + \int_c^b f$ (additivity)
> $f \le g \Rightarrow \int f \le \int g$ · $|\int f| \le \int|f|$

Symmetry tricks: on [−a,a] even ⇒ $2\int_0^a$, odd ⇒ 0; $\int_0^\pi x f(\sin x)dx = (\pi/2)\int_0^\pi f(\sin x)dx$.

Why unbounded fails (Darboux): if f is unbounded above on some subinterval, the sup there is +∞, so U(f,P)=+∞ for every P and U−L can't be made < ε ⇒ not integrable (e.g. 1/x on (0,1]). ∎

10 · FTC · both parts

PROVED ⚠

$F(x)=\int_a^x f$ is continuous when f integrable (|f|≤C + triangle inequality + squeeze).

FTC I. f cts ⇒ $F(x)=\int_a^x f$ is differentiable, F'(x)=f(x). *Proof:* $[F(x+h)-F(x)]/h = (1/h)\int_x^{x+h} f$; by EVT/IVT = f(ξ), ξ∈[x,x+h], and ξ→x as h→0 gives f(x). ∎

FTC II. G any primitive (G'=f) ⇒ $\int_a^b f = G(b)-G(a)$. *Proof:* F−G has zero derivative ⇒ constant. ∎

> **LEIBNIZ (VARIABLE LIMITS)**
> $\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f = f(g_2)g_2' - f(g_1)g_1'$

From FTC: parts $\int fg' = [fg] - \int f'g$; substitution $\int f(g)g' = \int_{g(a)}^{g(b)} f(u)du$; partial fractions for rationals ($\int dt/(1+t^2)=\arctan t$).

Worked Leibniz: $H(x)=\int_x^{x^2} e^{t^2} dt \Rightarrow H'(x) = e^{x^4}\cdot 2x - e^{x^2}\cdot 1$. **Improper:** $\int_0^1 dx/x$ diverges (unbounded at 0, handled as $\lim_{\varepsilon\to 0^+}\int_\varepsilon^1$).

Partial fractions: divide first if deg(num) ≥ deg(den); repeated/irreducible-quadratic factors → $A/(x-r) + (Bx+C)/(x^2+px+q) + \dots$. Recognise $\int dt/(1+t^2)=\arctan t$ and $\int dt/t=\ln|t|$.

Integration by parts and substitution both descend from the product and chain rules via FTC II — so every technique is, at heart, a theorem you can prove.

The integral function $F(x)=\int_a^x f$ is the bridge: continuous always, differentiable when f is continuous (FTC I), and its endpoint values give every definite integral (FTC II).

Proof Recap

SIDE 1

> ε-δ: ∀ε ∃δ, $0<|x-a|<\delta \Rightarrow |f-L|<\varepsilon$
> sup: (1) upper bound (2) least — BOTH
> Rolle→Cauchy MVT→MVT→L'Hôpital
> $R_n = f^{(n+1)}(c)/(n+1)!\cdot(x-x_0)^{n+1}$
> FTC: F'=f · $\int_a^b f = G(b)-G(a)$

Write proofs in full sentences; state the definition, name the technique, justify each step.

Quick proof-chain map: Completeness (LUB) ⇒ IVT & EVT ⇒ Rolle ⇒ Cauchy MVT ⇒ MVT ⇒ {monotonicity, L'Hôpital, Lagrange remainder}; EVT+IVT ⇒ FTC I ⇒ FTC II. Knowing the arrows lets you rebuild any proof from the one before it.

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flip → for side 2 · complex numbers, linear algebra & eigenvalues

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EXAM REVISION
Sem 1 2026 · **SIDE 2 OF 2**
Complex · linear algebra · prove it

`SIDE 2/2` STRUCTURE · Complex numbers · de Moivre & roots · Vectors & Cauchy-Schwarz · Gaussian elimination · Vector spaces · Rank-nullity · Eigenvalues

REVISION SHEET · ALL TOPICS

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11 · Complex Numbers

CARTESIAN → POLAR

$z = x+iy$, $i^2=-1$. Argand plot (x,y). Conjugate $\bar z = x-iy$; $\operatorname{Re} z = (z+\bar z)/2$, $\operatorname{Im} z = (z-\bar z)/2i$; $|z|^2 = z\bar z$.

POLAR / EXPONENTIAL

$$z = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$
$$r = |z| = \sqrt{x^2+y^2},\quad \theta = \arg z$$

arg z mod 2π; **principal Arg z ∈ (−π,π]**. Euler: $e^{i\theta} = \cos\theta + i\sin\theta$. Product/quotient: multiply/divide moduli, add/subtract args.

⚠ Cartesian↔polar is where sign/quadrant errors cost marks — check which quadrant (x,y) sits in before fixing θ.

Arithmetic: $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$; divide by multiplying by the conjugate, $z/w = z\bar w/|w|^2$.
Quadratics over ℂ: usual formula; for real coefficients complex roots come in conjugate pairs.

11b · de Moivre & Roots

INDUCTION PROOF ⚠

DE MOIVRE, N∈ℤ

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$
$$\text{i.e. } (r e^{i\theta})^n = r^n e^{in\theta}$$

Proof for n∈ℕ by induction: n=1 trivial; assume for k, then $(e^{i\theta})^{k+1} = (e^{i\theta})^k\cdot e^{i\theta} = e^{ik\theta}e^{i\theta} = e^{i(k+1)\theta}$ by the angle-sum identities. ∎ (extend to ℤ via z^{-1}.)

Use: multiple-angle formulas; expand $(1+ic)^n$ by binomial, split Re/Im.

N-TH ROOTS OF Z = R E^{IΘ}

$$a_k = r^{1/n} e^{i(\theta+2\pi k)/n},\quad k=0,\dots,n-1$$

n-th roots of unity: $e^{2\pi ik/n}$ — n equally-spaced points, a regular n-gon on the unit circle. ⚠ For $z^n=a$ give all n roots. Real-coeff polynomials: complex roots in conjugate pairs.

11c · Worked · Roots of Unity

SOLVE $Z^n=A$

Solve $z^3 = 8$. Write $8 = 8e^{i\cdot 0}$. Roots $a_k = 8^{1/3} e^{i(2\pi k)/3} = 2e^{2\pi ik/3}$, k=0,1,2: i.e. 2, $2e^{2\pi i/3}=-1+i\sqrt3$, $2e^{4\pi i/3}=-1-i\sqrt3$. Three points 120° apart, radius 2. ∎

de Moivre use: $(1+i)^8$ — polar $1+i = \sqrt2\, e^{i\pi/4}$, so $(1+i)^8 = (\sqrt2)^8 e^{i2\pi} = 16$. Cartesian expansion would be far slower. Assignment 1 used $(1+ic)^5\in\mathbb R \Rightarrow \sin 5\theta = 0$ to deduce $\tan(\pi/5)$.

12 · Vectors in $\mathbb R^n$

OPS + LENGTH

u+v, cu componentwise; $\|cv\| = |c|\|v\|$. Addition is head-to-tail; $A=(a_1,\dots,a_n)$ ↔ position vector $\overrightarrow{OA}$.

Length $\|v\| = \sqrt{v_1^2+\dots+v_n^2}$; unit vector $v/\|v\|$; standard basis e_i (1 in slot i).

Linear combination: $v = c_1v_1+\dots+c_kv_k$. **Dot product** $u\cdot v = \Sigma u_iv_i$; symmetric, bilinear, $u\cdot u \ge 0$ ($=0 \iff u=0$), $\|u\|=\sqrt{u\cdot u}$.
The dot-product axioms (symmetry, additivity, $(cu)\cdot v=c(u\cdot v)$, positive-definiteness) are exactly the **inner-product axioms** — the abstract setting in which Cauchy–Schwarz and the triangle inequality are proved once and reused.
⚠ $\mathbb R^n$ with these operations satisfies the 8 vector-space axioms — the prototype against which every abstract vector space on side 2 is checked.

Geometrically: c>0 keeps direction, c<0 reverses it, and $\|cu\|$ scales the length by |c| — the algebra and the picture must agree.

13 · Cauchy–Schwarz

THE KEY PROOF ⚠

CAUCHY–SCHWARZ

$$|u\cdot v| \le \|u\|\,\|v\|$$

Proof: for t∈ℝ, $p(t)=(tu+v)\cdot(tu+v)=\|u\|^2t^2 + 2(u\cdot v)t + \|v\|^2 \ge 0$. A non-negative quadratic has discriminant ≤ 0: $4(u\cdot v)^2 - 4\|u\|^2\|v\|^2 \le 0 \Rightarrow |u\cdot v| \le \|u\|\|v\|$. ∎

TRIANGLE INEQUALITY ⚠

$$\|u+v\| \le \|u\| + \|v\|$$

Proof: expand $\|u+v\|^2 = \|u\|^2+2(u\cdot v)+\|v\|^2 \le \|u\|^2+2\|u\|\|v\|+\|v\|^2 = (\|u\|+\|v\|)^2$ by Cauchy–Schwarz. ∎

Geometric: $u\cdot v = \|u\|\|v\|\cos\theta$ (proved by the cosine rule on $\|u-v\|^2$). $u\perp v \iff u\cdot v=0$.

Equality in Cauchy–Schwarz holds ⟺ u,v are parallel (the quadratic p(t) then has a real double root). This is the geometric content: cos θ = ±1.

13b · Projection & Cross

$\mathbb R^3$

PROJECTION OF V ONTO U

$$\text{proj}_u v = (u\cdot v)/(u\cdot u)\cdot u = (u\cdot v)/\|u\|^2\cdot u$$

Decompose $v = \text{proj}_u v + (v - \text{proj}_u v)$ (parallel + orthogonal).

CROSS PRODUCT ($\mathbb R^3$)

$$u\times v = [u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1]$$

Anticommutative $u\times v=-(v\times u)$; $u\times u=0$; ⊥ both u and v; right-hand rule; distributive; $c(u\times v)=(cu)\times v$. $\|u\times v\| = \|u\|\|v\|\sin\theta$ (via Lagrange identity $\|u\|^2\|v\|^2 = \|u\times v\|^2 + (u\cdot v)^2$).

Area of parallelogram = $\|u\times v\|$; triangle = ½$\|u\times v\|$; parallelepiped volume = $|u\cdot(v\times w)|$ (scalar triple product). Three vectors are coplanar ⟺ the triple product = 0.

14 · Lines & Planes

FORMS

Line: vector $x = R + t d$; normal $n\cdot(x-R)=0 \Rightarrow ax+by=c$.
Plane ($\mathbb R^3$): $x = R + s u + t v$; normal $n\cdot(x-R)=0$ ⊠ $ax+by+cz=d$; from 3 points $n = \overrightarrow{PQ} \times \overrightarrow{PR}$.

Distance point→line = $\|\overrightarrow{PQ} \times d\|/\|d\|$; point→plane = $|ax_0+by_0+cz_0-d|/\sqrt{a^2+b^2+c^2}$. ⚠ Equations are *not* unique.

Worked plane: through P(1,0,0), Q(0,1,0), R(0,0,1): $n = \overrightarrow{PQ}\times\overrightarrow{PR} = [-1,1,0]\times[-1,0,1] = [1,1,1]$, so plane is x+y+z = 1. Distance of origin to it = $|-1|/\sqrt3 = 1/\sqrt3$.

Worked line distance: point Q=(1,1,1) to the line through R=(0,0,0) with direction d=[1,0,0]: $\overrightarrow{RQ}=[1,1,1]$, $\overrightarrow{RQ}\times d=[0,1,-1]$, $\|\cdot\|=\sqrt2$, $/\|d\|=1$ ⊠ distance $\sqrt2$.
Equivalently, distance = ‖perpendicular component of the point-to-line vector‖ — the cross-product formula just packages that. Both routes earn full marks.

For a plane, the same idea gives distance = ‖projection of the point-to-plane vector onto the normal n‖.

The scalar triple product also tests coplanarity and gives signed volume — its sign encodes orientation (right- vs left-handed frame).

14b · Worked · Cauchy–Schwarz

NUMBERS

u=[1,2,2], v=[2,−1,2]: $u\cdot v = 2-2+4 = 4$; $\|u\|=3$, $\|v\|=3$, so $|u\cdot v|=4 \le 9 = \|u\|\|v\|$ ✓. Angle cos θ = 4/9. **proj_u v** = (4/9)[1,2,2].

Cross product: $u\times v = [2\cdot2-2\cdot(-1),\ 2\cdot2-1\cdot2,\ 1\cdot(-1)-2\cdot2] = [6,2,-5]$. Check $u\cdot(u\times v) = 6+4-10 = 0$ ✓ (orthogonal). $\|u\times v\| = \sqrt{65}$ = area of the parallelogram on u,v.

Orthogonal to a given vector in $\mathbb R^n$: swap two components, negate one, zero the rest — e.g. ⊥ to [a,b,…] is [−b,a,0,…]. Parallel: u = cv for some scalar c.

15 · Gaussian Elimination

SOLVE · RANK

Augmented [A | b]. **EROs** (preserve the solution set): swap $R_i\leftrightarrow R_j$; scale $R_i\to cR_i$ (c≠0); add $R_i\to R_i+cR_j$.

REF: zero rows at bottom, each leading entry right of the one above. **RREF** (Gauss–Jordan): leading 1s with zeros above & below. Back-substitute; free parameters for non-pivot columns. A **homogeneous** system (b=0) is always consistent (x=0 works) and has nontrivial solutions ⟺ a free variable exists.

TRICHOTOMY ⚠

every system: no solution / unique / ∞ many

Proof idea: two distinct solutions ⇒ their difference solves Ax=0 ⇒ adding scalar multiples gives ∞ many.
Inconsistent ⟺ a row [0 … 0 | c], c≠0. **Rank** = # nonzero rows in REF; # free params = n − rank[A|b].

Worked: x+y+z=6, 2y+z=7, z=3 (already echelon) ⇒ back-substitute z=3, then y=2, then x=1. Unique solution (rank 3 = 3 vars). A row like [0 0 0 | 5] would mark it inconsistent; a non-pivot column would give a free parameter and ∞ solutions.

Matrix product $(AB)_{ij}$ = (row i of A)·(col j of B); A acts as a function L_A(x)=Ax, and AB = composition L_A∘L_B. Not commutative (AB≠BA in general). Transpose reverses: $(AB)^T=B^TA^T$.

16 · Inverses & Determinants

N×N

A invertible if ∃B: AB=I=BA. **Inverse unique** ⚠ : B = BI = B(AC) = (BA)C = C. Compute via $[A\,|\,I] \to [I\,|\,A^{-1}]$.

2×2

$$A^{-1} = 1/(ad-bc)\cdot[d\ -b;\ -c\ a],\quad \det=ad-bc$$

$(AB)^{-1} = B^{-1}A^{-1}$ ⚠ ; $(A^T)^{-1}=(A^{-1})^T$. **Cofactor (Laplace) expansion:** $\det A = \Sigma_k a_{ik}C_{ik}$, $C_{ik}=(-1)^{i+k}\cdot$(minor). Triangular ⇒ det = product of diagonal.

A invertible ⟺ det A ≠ 0; det(AB)=det A·det B; |det A| = volume of the parallelepiped of the rows.

16b · Invertibility TFAE

CENTRAL ⚠

For A (n×n) the following are **equivalent**: (1) A invertible; (2) Ax=b has a unique solution ∀b; (3) Ax=0 ⇒ x=0; (4) RREF(A)=I_n; (5) A is a product of elementary matrices; (6) rank A = n; (7) nullity A = 0; (8) columns are a basis of $\mathbb R^n$; (9) det A ≠ 0. Proved by the cycle (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(1).

16c · Worked · Det & Eigen

3×3 / 2×2

3×3 det (expand row 1): det[1 2 3; 0 1 4; 5 6 0] = $1(1\cdot0-4\cdot6) - 2(0\cdot0-4\cdot5) + 3(0\cdot6-1\cdot5) = -24 + 40 - 15 = 1$ ⇒ invertible.

2×2 eigenvalues: A=[2 1; 1 2]. $\chi(\lambda)=(2-\lambda)^2-1 = \lambda^2-4\lambda+3 = (\lambda-1)(\lambda-3)$. λ=1 ⇒ eigenvector [1,−1]; λ=3 ⇒ [1,1]. Two distinct λ ⇒ diagonalisable.

Elementary matrices: E = one ERO applied to I; doing an ERO on A = left-multiplying by E, each E invertible (reverse ERO). One-sided inverse suffices: BA=I ⇒ A invertible with A^{-1}=B. $(AB)^{-1}=B^{-1}A^{-1}$ reverses order.

Worked 2×2 inverse: A=[1 2; 3 4], det = 4−6 = −2 ≠ 0 ⇒ invertible; A^{-1} = (−1/2)[4 −2; −3 1] = [−2 1; 1.5 −0.5]. Check AA^{-1} = I.
The cofactor expansion can be taken along *any* row or column — pick the one with the most zeros to cut work. Row-reduce to triangular and multiply the diagonal for larger n.
det(AB)=det A·det B gives $\det(A^{-1})$=1/det A and $\det(A^T)$=det A — handy checks when a computed inverse or eigenvalue looks wrong.

17 · Vector Spaces

8 AXIOMS ⚠

A **vector space over** ℝ is a set V with + and scalar mult (closed), a zero 0, inverses −u, satisfying 8 axioms: commutativity & associativity of +; 0; −u; c(u+v)=cu+cv; (c+d)u=cu+du; c(du)=(cd)u; 1u=u.

⚠ To PROVE V is a vector space: verify closure + every axiom; to disprove, exhibit one failing axiom.
Examples beyond $\mathbb R^n$: functions f:ℝ→ℝ (pointwise); (0,∞) with x⊕y=xy, c⊙x=x^c (zero is 1).

An **isomorphism** is a bijective linear map; isomorphic spaces share dimension and all structural facts. Assignment 1 asked to verify a recurrence-defined function space is isomorphic to $\mathbb R^2$ (a function is determined by f(0),f(1)).

To prove a subset is a subspace it suffices to check the three subspace conditions; you get the other axioms free because they are inherited from the ambient space.

17b · Subspace Test

3 CONDITIONS ⚠

S⊆ℝⁿ IS A SUBSPACE IF

(1) 0 ∈ S
(2) closed under + (u,v∈S ⇒ u+v∈S)
(3) closed under scalar mult (u∈S ⇒ cu∈S)

Then S is itself a vector space. Lines/planes through the origin are subspaces; {x+y≥0} or {xy=0} are not.
span(S) is always a subspace (proof: closure).
Subspaces of $\mathbb R^3$: {0}, lines, planes through 0, $\mathbb R^3$.

18 · Span · Independence

BASIS

Span(S) = $\{c_1v_1+\dots+c_kv_k\}$. b ∈ span($v_1\dots v_k$) ⟺ [$v_1\dots v_k$ | b] consistent.

LINEAR INDEPENDENCE

$$c_1v_1+\dots+c_kv_k = 0 \Rightarrow c_1=\dots=c_k=0$$
(only the trivial solution)

Otherwise dependent (any set containing 0 is dependent). ⚠ n vectors in $\mathbb R^n$ l.i. ⟺ [$v_1\dots v_n$] invertible; k l.i. vectors in $\mathbb R^n$ ⇒ k ≤ n.

Basis of S = spans S *and* linearly independent. **Basis Theorem:** any two bases have the same size = **dim S** (dim $\mathbb R^n$ = n).

S spans $\mathbb R^n$ ⇒ |S| ≥ n; S l.i. in $\mathbb R^n$ ⇒ |S| ≤ n. Corollary: exactly n vectors form a basis ⟺ they are l.i. ⟺ they span. b∈span($v_1\dots v_k$) ⟺ the augmented system [$v_1\dots v_k$ | b] is consistent.

18b · Worked · Independence

TRIVIAL-ONLY TEST

Are v_1=[1,2,1], v_2=[2,1,0], v_3=[1,−1,−1] l.i.? Row-reduce [v_1 v_2 v_3] (as columns): det = $1(1\cdot-1-0\cdot-1) - 2(2\cdot-1-0\cdot1) + 1(2\cdot-1-1\cdot1) = -1 + 4 - 3 = 0$ ⇒ dependent. Indeed $v_3 = v_2 - v_1$.

⚠ A zero determinant / a free column means the only-trivial-solution test fails — there is a nontrivial dependence relation.

18c · Worked · Subspace?

CHECK 3 CONDITIONS

Is W = {(x,y,z): x+y+z=0} a subspace of $\mathbb R^3$? 0=(0,0,0) ✓; sum of two solutions solves it ✓; scalar multiple solves it ✓ ⇒ **yes** (a plane through 0, dim 2). But {x+y+z=1} fails (0∉it), and {xy=0} fails closure under +.

Basis of W above: solve x=−y−z ⇒ vectors [−1,1,0], [−1,0,1] span and are l.i. ⇒ dim W = 2. The matrix-space $M_{m\times n}(\mathbb R)$ and the polynomial space P_n are also vector spaces (the abstract examples examined in Assignment 1).

19 · Rank–Nullity

THE THEOREM ⚠

For A (m×n): **row(A)**⊆$\mathbb R^n$, **col(A)**⊆$\mathbb R^m$, **null(A)**={x: Ax=0} ⊆$\mathbb R^n$ — all subspaces (proof for null: closure).

Finding bases (R = RREF A): row space → nonzero rows of R; column space → columns of A in R's *pivot* positions (col(A)≠col(R) but same dim); null space → solve Rx=0, separate parameters.

RANK–NULLITY ⚠

$$\text{rank}(A) + \text{nullity}(A) = n$$

rank = # leading 1s = dim row = dim col; nullity = dim null. *Proof:* the n columns split into pivot (→ rank) vs free (→ nullity). Corollary: in dim-n space, n vectors are l.i. ⟺ they span.

20 · Linear Transformations

STANDARD MATRIX

T:$\mathbb R^n\to\mathbb R^m$ **linear** if T(u+v)=T(u)+T(v) and T(cu)=cT(u). Hence T(0)=0.

Standard-matrix thm ⚠ : every linear T = L_A with A = [T(e_1) … T(e_n)] (m×n, unique). Proof: x=Σx_ie_i + linearity. Composition: A_{S∘T} = A_S A_T.

⚠ **Prove T NOT linear:** exhibit a failing case — absolute values, constants or squares break it (e.g. T with a |z| term fails T(cu)=cT(u) at c=−1). T:$\mathbb R^n\to\mathbb R^n$ bijective ⟺ A_T invertible.

21 · Eigenvalues

AX = ΛX

Nonzero x, scalar λ with Ax=λx: λ **eigenvalue**, x **eigenvector**. **Eigenspace** E_λ = null(A−λI) (a subspace; 0∈E_λ but is never an eigenvector).

CHARACTERISTIC POLYNOMIAL

$$\lambda \text{ eigenvalue} \iff \chi_A(\lambda)=\det(A-\lambda I)=0$$

Method: solve det(A−λI)=0 for λ, then for each λ find null(A−λI). ⚠ Eigenvectors must be nonzero; report a basis of each eigenspace. Real A can have complex eigenvalues (rotation) — links to the complex stream.

21b · Worked · Rank–Nullity

CHECK N

A is 3×4 with RREF having pivots in columns 1,2 (2 pivots). Then rank = 2, and nullity = 4 − 2 = 2 (two free variables). Column space dim 2 ⊆ $\mathbb R^3$; null space dim 2 ⊆ $\mathbb R^4$. Sum 2+2 = 4 = n ✓.

Standard matrices: rotation by θ = [cos θ −sin θ; sin θ cos θ]; $R_\alpha R_\beta = R_{\alpha+\beta}$ recovers the angle-sum identities. Reflection, projection, scaling each have their own A.

21c · The Multiplicity Trap

WHEN DIAG FAILS

A=[2 1; 0 2]: $\chi(\lambda)=(2-\lambda)^2$, so λ=2 with **algebraic mult 2**. But A−2I = [0 1; 0 0] has nullity 1 ⇒ **geometric mult 1** < 2. So A is NOT diagonalisable — only one independent eigenvector.

Contrast A=[2 0; 0 2] (=2I): same λ=2 twice but every vector is an eigenvector, geometric mult 2 ⇒ already diagonal.

Eigenvectors for distinct λ are l.i. — the lemma behind "n distinct eigenvalues ⇒ diagonalisable".
Trace = Σλ and det = Πλ give fast sanity checks on a computed spectrum.

A real matrix can have complex eigenvalues (a rotation [cos θ −sin θ; sin θ cos θ] has $\lambda = e^{\pm i\theta}$) — this is exactly where the complex-number stream feeds back into linear algebra.
Always compute $\chi_A(\lambda)$ and factor it *before* hunting eigenvectors — the marks for "eigenvalue reasoning" live in showing det(A−λI)=0, not just stating the answers.

22 · Diagonalisation

$A = PDP^{-1}$

A is **diagonalisable** if A=PDP^{-1} (D diagonal) ⟺ A has n linearly independent eigenvectors (columns of P), eigenvalues on D's diagonal.

Sufficient: n distinct eigenvalues ⇒ diagonalisable (eigenvectors for distinct λ are l.i.). **Use:** $A^k = PD^kP^{-1}$; Markov chains (e.g. PageRank).

⚠ **The multiplicity trap:** a repeated eigenvalue *may* still diagonalise — check geometric mult (dim E_λ) = algebraic mult (repetition). If any eigenspace is too small, A is **not** diagonalisable.
Powers worked: with A=[2 1;1 2] (λ=1,3; P=[1 1;−1 1]), $A^k = P\,\text{diag}(1,3^k)\,P^{-1}$. Markov chains: the steady state is the eigenvector for λ=1, found as 3^k-terms wash out under repeated multiplication.
Diagonalising also solves linear recurrences and decouples systems of ODEs — same machine: change to the eigenbasis, act diagonally, change back.

Final reflex: every "compute" question in 1961 hides a "justify" — name the theorem you are invoking and you bank the method marks even if arithmetic slips.

23 · Proof Pattern Belt

PICK THE RECIPE

- **"Prove ∀..."** → arbitrary element OR induction (strong if order ≥2).
- **"Prove a limit"** → ε–δ: fix ε, bound, pick δ.
- **"Prove c = sup"** → both clauses; clause 2 via c−ε.
- **"∃ a root / value"** → IVT (sign change / limits ±∞).
- **"How many roots"** → MVT monotonicity + IVT.
- **"Bound the error"** → Lagrange remainder, solve for n.
- **"Is it a subspace / VS"** → check 0 + closure / 8 axioms.
- **"l.i.?"** → trivial-solution-only of $\Sigma c_iv_i=0$.
- **"Diagonalisable?"** → n l.i. eigenvectors / geom = alg mult.
- **"Prove an inequality"** → Cauchy–Schwarz, or MVT, or a non-negative square.
- **"Find all roots of z^n=a"** → polar form + the n-root formula.
- **"Disprove"** → one counterexample suffices.

24 · High-Yield Traps

WHERE RIGOUR WINS

- ★ sup proof needs **both** clauses, not just an upper bound.
- ★ Limit DNE needs **two explicit sequences**.
- ★ EVT needs **closed + bounded + cts**.
- ★ Bounded ⇏ integrable (Dirichlet).
- ★ Taylor *series* reps f only if $R_n\to0$.
- ★ Critical point ⇏ extremum (x^3).
- ★ Repeated eigenvalue ⇒ check geom = alg mult.
- ★ col(A) uses A's columns, not RREF's.
- ★ Eigenvector must be **nonzero**; report a basis of E_λ.
- ★ z^n=a has **all n** roots — don't stop at one.
- ★ Proofs in full sentences (LO2), not symbol-soup.

SIA → ***Method marks survive a slipped number. State the definition, name the technique, write full sentences. That is the whole difference between 1961 and 1061.***

LA Recap

SIDE 2

$$z = re^{i\theta}\ \cdot\ (re^{i\theta})^n = r^ne^{in\theta}$$
$$|u\cdot v| \le \|u\|\|v\|\ \cdot\ \text{proj} = (u\cdot v)/\|u\|^2\cdot u$$
$$\text{rank} + \text{nullity} = n\ \cdot\ \det\ne0 \iff \text{invertible}$$
$$Ax=\lambda x\ \cdot\ \chi_A(\lambda)=\det(A-\lambda I)=0$$
$$\text{diag} \iff n \text{ l.i. eigenvectors}$$