0 · Exam Blueprint
READ FIRST

★ MATH1061 runs **two parallel streams**: this side is Calculus (limits → derivatives → integrals → series); flip for Linear Algebra. Assessment: weekly quizzes 8% · A1 5% · A2 10% · in-person Quiz A 15% · tutorials 2% · **final exam 60%**.

Most-tested moves: 0/0 limits (factor or L'Hôpital); differentiate with chain + product/quotient; classify critical points; evaluate a definite integral via FTC + a technique (sub / parts / partial fractions); write a Maclaurin series and use it.

Method marks: show the working — state the rule, then the substitution, then the answer. A dropped chain-rule factor or a missed +C is the standard mark-loss.

SIA → *Two reflexes: name the form before you compute (is it 0/0? is it a product?), and always check the hypotheses – L'Hôpital needs 0/0 or ∞/∞, FTC needs continuity.*

1 · Functions & Limits
WK 1–2

Function f:A→B, one output per input; range = image ⊆ codomain. **Injective** (1-1), **surjective** (onto, range = codomain), **bijective** = both ⇒ f^{-1} exists. **Composition** (g∘f)(x)=g(f(x)).

Limit. $\lim_{x\to a} f(x)=L$: f(x) is forced arbitrarily close to L by taking x close to (≠) a. Two-sided limit exists *iff* both one-sided limits exist and agree.

LIMIT LAWS (IF BOTH LIMITS EXIST)

ALGEBRA OF LIMITS
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lim(kf)=k·lim f · lim(f±g)=lim f ± lim g
lim(fg)=(lim f)(lim g)
lim(f/g)=lim f / lim g  only if lim g≠0
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If f is continuous at a, $\lim_{x\to a} f(x)=f(a)$ — so most limits are "plug in"; only the *joints* of piecewise functions need care. Never use the quotient law when the denominator limit is 0 — factor, rationalise or use L'Hôpital instead.

1b · Squeeze & Standard Limits
★ MEMORISE

SQUEEZE (SANDWICH)
g≤f≤h near a, lim g = lim h = L ⇒ lim f = L

STANDARD LIMITS
$\lim_{x\to 0} \sin x / x = 1$
$\lim_{x\to 0} (1-\cos x)/x = 0$
$\lim_{x\to\infty} (1+1/x)^x = e$

Classic squeeze: $-|x| \le x\cdot\sin(1/x) \le |x| \Rightarrow \lim_{x\to 0} x \sin(1/x)=0$.

1c · 0/0 Limits · Worked
FACTOR FIRST

$\lim_{x\to 1} (x^2-1)/(x^2+x-2)$:
= (x-1)(x+1) / (x-1)(x+2)
= (x+1)/(x+2) → 2/3

Cancel the common factor causing the 0; the limit can exist even when f(1) is undefined.

Rationalise. $\lim_{x\to 0} (\sqrt{1+x}-1)/x \cdot (\sqrt{1+x}+1)/(\sqrt{1+x}+1) = 1/(\sqrt{1+x}+1) \to 1/2$.

At infinity. Divide by the highest power: $\lim_{x\to\infty} (3x^2+1)/(2x^2-x) = 3/2$ (horizontal asymptote).

2 · Continuity & IVT
WK 3

f is **continuous at a** iff $\lim_{x\to a} f(x)=f(a)$ (all three: limit exists, f(a) defined, equal). Polynomials, roots, e^x, ln, trig are continuous on their domains; sums/products/quotients/compositions stay continuous.

INTERMEDIATE VALUE THEOREM
f continuous on [a,b], N between f(a) & f(b)
⇒ ∃ c∈[a,b] with f(c)=N

Use IVT to show a **root exists**: f(a)<0<f(b) ⇒ some c with f(c)=0. Needs a *closed* interval; gives existence, not the value.

Inverse & hyperbolic. Restrict the domain so f is injective *before* inverting; f^{-1} is the reflection of f in y=x.

Key examples: $\cosh x=(e^x+e^{-x})/2$, $\sinh x=(e^x-e^{-x})/2$, with $\cosh^2 x - \sinh^2 x = 1$.

3 · The Derivative
WK 3–4

DEFINITION
$f'(a)=\lim_{h\to 0} [f(a+h)-f(a)] / h$
= slope of tangent at (a,f(a))

Tangent line: $y = f(a) + f'(a)(x-a)$. Differentiable ⇒ continuous (not conversely — |x| has a corner at 0).

RULES

LINEARITY / PRODUCT / QUOTIENT / CHAIN
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(kf)'=kf' · (f±g)'=f'±g'
(fg)' = f'g + fg'   (product)
(f/g)' = (f'g - fg')/g²   (quotient)
(g∘f)'(x) = g'(f(x))·f'(x)   (chain)
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Implicit: differentiate F(x,y)=0 in x, chain-rule the y-terms, solve y′. e.g. $x^2+y^2=1 \Rightarrow 2x+2yy'=0 \Rightarrow y'=-x/y$.

Logarithmic: $y=x^x \Rightarrow \ln y = x \ln x \Rightarrow y'/y = \ln x + 1 \Rightarrow y' = x^x(\ln x + 1)$.

Quotient worked. d/dx (sin x / x) = (x cos x − sin x)/x^2.
Chain worked. d/dx $\sin(x^2) = \cos(x^2)\cdot 2x$.

Higher derivatives f″, f‴ feed the second-derivative test and Taylor coefficients; for a product use the product rule repeatedly (or Leibniz's formula). Differentiable ⇒ continuous, but not the reverse (corners and cusps).

4 · Standard Derivatives
D/DX · ★ TABLE

F(X)	F′(X)
x^k	$k x^{k-1}$
e^x	e^x
a^x	$a^x \ln a$
ln x	1/x
sin x	cos x
cos x	−sin x
tan x	$\sec^2 x$
sec x	sec x tan x
cot x	$-\csc^2 x$
csc x	−csc x cot x
$\sin^{-1} x$	$1/\sqrt{1-x^2}$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$
$\tan^{-1} x$	$1/(1+x^2)$
sinh x	cosh x
cosh x	sinh x

$\cosh^2 x - \sinh^2 x = 1$. Combine with the chain rule for any composite, e.g. d/dx ln(f(x)) = f′(x)/f(x).

5 · L'Hôpital's Rule
WK 5

INDETERMINATE 0/0 OR ∞/∞
$\lim_{x\to a} f/g = \lim_{x\to a} f'/g'$
(when the RHS exists)

Check the form first. Other forms: 0·∞ → rewrite as 0/0 or ∞/∞; for 1^∞, 0^0, ∞^0 take logs, then apply.

Worked. $\lim_{x\to 0} (e^x-1-x)/x^2$ is 0/0 → $(e^x-1)/2x$ still 0/0 → $e^x/2 \to 1/2$.

0·∞ worked. $\lim_{x\to 0^+} x \ln x = \lim (\ln x)/(1/x) = (\infty/\infty) \to (1/x)/(-1/x^2) = -x \to$ **0**.

Warning: never apply L'Hôpital to a determinate form (e.g. 3/0 or 5/2) — re-check the form after every step.

6 · Extrema & Curve Sketching
WK 5–6

FIRST DERIVATIVE TEST
- f′>0 ⇒ increasing; f′<0 ⇒ decreasing
- **Critical point** f′(c)=0: necessary, *not* sufficient (x^3 at 0)
- f′: −→+ at c ⇒ local **min**; +→− ⇒ local **max**

SECOND DERIVATIVE TEST
- f″>0 ⇒ concave up; f″<0 ⇒ concave down
- f′(c)=0, f″(c)>0 ⇒ min; f″(c)<0 ⇒ max
- **Inflection**: concavity changes (f″=0 necessary, check sign change)

Sketch checklist: domain · intercepts · sign of f′ (incr/decr, crit pts) · sign of f″ (concavity, inflections) · end behaviour x→±∞.

6b · Optimisation · Worked
METHOD MARKS

Method: write the quantity & its domain → f′(x)=0 for critical points → classify (1st/2nd test) → compare with **endpoints** for the global optimum.

Eg. max area of a rectangle of perimeter 20: A=x(10−x), A′=10−2x=0 ⇒ x=5; A″=−2<0 ⇒ max. A=25 (a square).
Trap: f″(c)=0 does not force an inflection; never forget endpoint values on a closed interval.

7 · Riemann Integral
WK 9

DEFINITE INTEGRAL = SIGNED AREA
$\int_a^b f\,dx = \lim_{N\to\infty} \Sigma_k f(x_k^*)\Delta x$
Δx = (b-a)/N

Exists for continuous f. Lower/upper sums $L_N \le \int \le U_N$ bound it (handy for monotonic f). Sub-intervals partition $a=x_0<x_1<\ldots<x_N=b$; the sample point $x_k^* \in [x_{k-1},x_k]$.

BASIC PROPERTIES
$\int_a^a f = 0 \cdot \int_a^b f = -\int_b^a f$
$\int_a^b(\alpha f+\beta g) = \alpha\int f + \beta\int g$ (linearity)
$\int_a^c f + \int_c^b f = \int_a^b f$

Signed area: regions below the x-axis count negative — split at the zeros if you want true geometric area.

7b · Curve Sketch · Worked
F=X³-3X

$f'=3x^2-3=0 \Rightarrow x=\pm1$; f″=6x. x=−1: f″<0 ⇒ local max f=2; x=1: f″>0 ⇒ local min f=−2. Inflection at x=0. Odd, roots 0,±√3, ends ±∞. The sign charts of f′ and f″ give the whole shape. On a closed interval, also test endpoints for the global extremum; a critical point alone need not be one.

8 · Fundamental Theorem
FTC · WK 10

FTC I & II
I: $d/dx \int_c^x f(t)\,dt = f(x)$
II: $\int_a^b F'(x)\,dx = F(b) - F(a)$

VARIABLE LIMITS (LEIBNIZ)
$d/dx \int_c^{g(x)} f(t)\,dt = f(g(x))\cdot g'(x)$
$d/dx \int_x^c f(t)\,dt = -f(x)$

Don't drop g′(x) on a variable upper limit; the sign flips for a variable *lower* limit.

FTC worked. $\int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1 - 0 = 1$. And $d/dx \int_0^{x^2} \sin t\,dt = \sin(x^2)\cdot 2x$.

Part I → II. Part I says $\int_c^x f$ is an antiderivative of f; Part II evaluates any antiderivative at the endpoints. Continuity of f on [a,b] is the hypothesis both need; FTC is what turns "antiderivative" into a number.

9 · Antiderivatives
∫ · ★ TABLE

F(X)	∫ F DX (+C)
x^n (n≠−1)	$x^{n+1}/(n+1)$
1/x	ln|x|
e^x	e^x
a^x	$a^x/\ln a$
sin x	−cos x
cos x	sin x
$\sec^2 x$	tan x
tan x	ln|sec x|
sec x	ln|sec x + tan x|
$1/\sqrt{1-x^2}$	$\sin^{-1} x$
$1/(1+x^2)$	$\tan^{-1} x$

Always +C on an indefinite integral.

10 · Substitution & Parts
WK 10–11

SUBSTITUTION (UNDO CHAIN)
$\int f(u(t))\cdot u'(t)\,dt = \int f(s)\,ds$, s=u(t)
change the limits in the definite case

BY PARTS (UNDO PRODUCT)
$\int u\,dv = uv - \int v\,du$
choose u by LIATE

LIATE = Log · Inverse-trig · Algebraic · Trig · Exp — pick the *earlier* type as u (its derivative simplifies). Eg $\int x e^x\,dx$: u=x, dv=e^xdx ⇒ $x e^x - e^x + C$.

ln by parts: ∫ ln x dx: u=ln x, dv=dx ⇒ x ln x − ∫ x·(1/x) dx = x ln x − x + C.

Sub worked: $\int 2x e^{x^2} dx$, s=x^2, ds=2x dx ⇒ $\int e^s ds = e^{x^2} + C$. For a definite integral, change the limits too (don't back-substitute).

Cyclic parts: $\int e^x \cos x\,dx$ — parts twice returns the original integral I; solve algebraically: I = ½e^x(sin x + cos x) + C.

Definite sub: $\int_0^1 2x(x^2+1)^3\,dx$, s=x^2+1 (1→2): $\int_1^2 s^3\,ds = [s^4/4]_1^2 = 15/4$.

x² by parts (twice): $\int x^2 e^x\,dx = x^2e^x - 2\int x e^x\,dx = x^2e^x - 2(xe^x - e^x) + C = e^x(x^2-2x+2) + C$. A polynomial × e^x/sin/cos: parts lowers the polynomial degree each pass.

11 · Partial Fractions & Trig Sub
RATIONAL + ROOTS

DISTINCT LINEAR FACTORS
(a+bx)/[(x-λ)(x-μ)] = A/(x-λ) + B/(x-μ)

Solve A,B by cover-up / equating coefficients. Eg 1/[(x+1)(x−1)] = (−½)/(x+1) + (½)/(x−1). *Reduce an improper fraction* (divide) before splitting.

TRIG INTEGRALS & SUBSTITUTION

$\int \sin^m x \cos^n x\,dx$: if a power is odd, peel one factor & use $\sin^2+\cos^2=1$; if both even, use $\sin^2 x=(1-\cos 2x)/2$, $\cos^2 x=(1+\cos 2x)/2$.

TRIG SUBSTITUTION
√(a²-x²) ⇒ x=a sinθ
√(a²+x²) ⇒ x=a tanθ
√(x²-a²) ⇒ x=a secθ

PF integral worked. ∫ dx/[(x+1)(x−1)] = ∫ (−½/(x+1) + ½/(x−1))dx = ½ ln|(x−1)/(x+1)| + C.

Trig-sub idea. $\int dx/\sqrt{1-x^2}$ with x=sinθ, dx=cosθ dθ ⇒ ∫ dθ = θ = $\sin^{-1} x$ + C (recovers the standard antiderivative). Repeated factors $(x-\lambda)^2$ need a $B/(x-\lambda)^2$ term too; an irreducible quadratic gets a (Cx+D)/(quadratic) term.

11b · Improper Integrals
WK 11–12

Defined as limits: $\int_a^\infty f = \lim_{b\to\infty} \int_a^b f$; singularity at c handled with a one-sided limit. Converges iff the limit is finite.

BENCHMARKS
$\int_1^\infty dx/x^p$ converges iff p>1
$\int_0^1 dx/x^p$ converges iff p<1

Worked. $\int_1^\infty dx/x^2 = \lim_{b\to\infty} [-1/x]_1^b = \lim (1 - 1/b) = 1$ (converges, p=2>1).
Trap: a singularity inside [a,b] must be split, and *both* pieces must converge for the whole integral to converge.

12 · Geometric Applications
WK 12

AREA BETWEEN CURVES (F≥G)
$A = \int_a^b [f(x) - g(x)]\,dx$ (top - bottom)

ARC LENGTH OF Y=F(X)
$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$

VOLUME OF REVOLUTION
about x-axis (disc): $V = \int_a^b \pi f(x)^2\,dx$
about y-axis (shell): $V = \int_a^b 2\pi x f(x)\,dx$

Trap: subtract in the right order; don't swap the disc (πf^2) and shell ($2\pi x f$) formulas.

Area worked. Between y=x and y=x^2 on [0,1]: x≥x^2 there, so $A = \int_0^1 (x-x^2)dx = [x^2/2 - x^3/3]_0^1 = 1/2 - 1/3 = 1/6$.

Volume worked. y=√x on [0,4] about the x-axis: $V = \int_0^4 \pi x\,dx = \pi[x^2/2]_0^4 = 8\pi$.

Arc length idea. For y=f(x), the element $ds=\sqrt{1+(f')^2}dx$ comes from Pythagoras on (dx, dy); integrate it for total length.

Find the intersection first. For area between curves, set f(x)=g(x) to get the limits a,b, then integrate top−bottom; if they cross inside, split the integral at the crossing. Sketch first to see which curve is on top.

13 · Taylor & Maclaurin
WK 7–8

TAYLOR POLYNOMIAL ABOUT A
$T_n(x) = \Sigma_{k=0}^n f^{(k)}(a)/k! \cdot (x-a)^k$
= f(a) + f′(a)(x-a) + f″(a)/2!·(x-a)² + …

a=0 ⇒ **Maclaurin.** T_n matches f and its first n derivatives at a.

REMAINDER R_N = F − T_N
Lagrange: $R_n = f^{(n+1)}(c)/(n+1)! \cdot (x-a)^{n+1}$
series equals f only where $R_n \to 0$

13b · Standard Series
★ KNOW THESE

MACLAURIN SERIES
$e^x = \Sigma\, x^k/k!$ (all x)
$\sin x = \Sigma (-1)^k x^{2k+1}/(2k+1)!$
$\cos x = \Sigma (-1)^k x^{2k}/(2k)!$
$\ln(1+x) = \Sigma (-1)^{k-1} x^k/k$ (|x|<1)
$(1+x)^\alpha = \Sigma\, C(\alpha,k)\, x^k$ (|x|<1)

GEOMETRIC
$1+x+\ldots+x^n = (1-x^{n+1})/(1-x)$
$\Sigma_{k\ge 0} x^k = 1/(1-x)$ (|x|<1)

Term-by-term trick: substitute / differentiate / integrate a known series rather than recompute derivatives.

From $\Sigma x^k=1/(1-x)$: differentiate ⇒ $\Sigma k x^{k-1} = 1/(1-x)^2$; integrate −x into 1/(1+x) ⇒ recovers ln(1+x).

Geometric series value: $\Sigma_{k\ge 0} ar^k = a/(1-r)$ for |r|<1. Eg $\Sigma (1/2)^k = 1/(1-½) = 2$; diverges for |r|≥1. The partial sum is $a(1-r^{n+1})/(1-r)$.

13c · Taylor · Worked
USE A KNOWN SERIES

Maclaurin of cos x to x^4: cos x ≈ 1 − x^2/2 + x^4/24.
So $\lim_{x\to 0} (1-\cos x)/x^2 = 1/2$ (the x^2/2 term dominates) — faster than two L'Hôpitals.

And $e^{x^2} = 1 + x^2 + x^4/2 + \ldots$ by substituting x^2 into e^x — no new derivatives needed. Integrate term-by-term ⇒ $\int e^{x^2}$ dx series (no elementary closed form). The first non-zero term controls the small-x behaviour.

About a≠0. ln x about a=1: f(1)=0, f′=1/x→1, f″=−1/x²→−1 ⇒ ln x ≈ (x−1) − $(x-1)^2/2 + (x-1)^3/3$ − … (matches ln(1+u), u=x−1). Build coefficients $f^{(k)}(a)/k!$ one derivative at a time.

Radius / convergence. e^x, sin, cos converge for *all* x; ln(1+x) and $(1+x)^\alpha$ only for |x|<1; geometric for |x|<1. Always state the radius before equating a series to f. Off-by-one in the factorial/power is the standard slip.

Calculus Formula Belt
SIDE 1

(g∘f)′=g′(f)·f′ · (f/g)′=(f′g-fg′)/g²
∫ u dv = uv − ∫ v du · LIATE
FTC: $\int_a^b F'=F(b)-F(a)$ · $d/dx\int_c^x f=f(x)$
L'Hôpital: 0/0,∞/∞ ⇒ lim f′/g′
$\Sigma x^k=1/(1-x)$ (|x|<1)

SIA → *When a limit/integral looks ugly, ask "is there a standard series or a substitution that turns it into something on this sheet?" – usually yes. State the rule, show the substitution, then the answer – that is where the method marks live.*

14 · Complex Numbers
WK 1–2

$i^2=-1$; $z=a+ib$, $\mathrm{Re}(z)=a$, $\mathrm{Im}(z)=b$ (both *real*). $\mathbb{R}\subset\mathbb{C}$.

ARITHMETIC
$(a+ib)+(c+id) = (a+c)+i(b+d)$
$(a+ib)(c+id) = (ac-bd)+i(ad+bc)$

Conjugate $\bar{z}=a-ib$: $z\bar{z} = a^2+b^2 \in \mathbb{R}$, and $\overline{zw} = \bar{z}\cdot\bar{w}$, $\overline{z+w} = \bar{z}+\bar{w}$.

DIVISION — REALISE THE DENOMINATOR
$w/z = w\bar{z} / (z\bar{z}) = w\bar{z} / |z|^2$

Equality: $a+ib = c+id \iff a=c$ and $b=d$.

Powers of i cycle: $i, -1, -i, 1, i, \ldots$ (period 4) — reduce the exponent mod 4.

Division worked. $(3+i)/(1-2i) \cdot (1+2i)/(1+2i) = (3+6i+i-2)/(1+4) = (1+7i)/5 =$ $1/5 + (7/5)i$.

Quadratic over $\mathbb{C}$. $z^2+z+1=0 \Rightarrow z = (-1\pm\sqrt{(-3)})/2 = -1/2 \pm (\sqrt{3}/2)i$ — a conjugate pair, the primitive cube roots of unity ($\neq 1$). The quadratic formula works over $\mathbb{C}$ with $\sqrt{}$ of a negative; the discriminant being negative is what forces complex roots.

15 · Modulus, Polar & Euler
ARGAND PLANE

Modulus $|z|=\sqrt{(a^2+b^2)}=\sqrt{(z\bar{z})}$ = distance from 0; $|z-w|$ = distance $z\leftrightarrow w$. $|zw|=|z||w|$, $|z/w|=|z|/|w|$, triangle ineq $|z+w|\leq|z|+|w|$.

Argument arg $z=\theta$ with $z=|z|(\cos\theta+i\sin\theta)$; principal Arg $z\in(-\pi,\pi]$. Read the quadrant from the diagram — not just $\tan^{-1}(b/a)$.

POLAR / EXPONENTIAL (EULER)
$e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow z = r\,e^{i\theta}$
$z_1z_2 = r_1r_2\,e^{i(\theta_1+\theta_2)}$
$z_1/z_2 = (r_1/r_2)\,e^{i(\theta_1-\theta_2)}$

Euler's identity: $e^{i\pi}+1=0$. Multiplying = multiply moduli, add arguments.

Useful identities: $\mathrm{Re}(z)=(z+\bar{z})/2$, $\mathrm{Im}(z)=(z-\bar{z})/2i$, $z^{-1} = \bar{z}/|z|^2$ ($|z|=1 \Rightarrow z^{-1}=\bar{z}$), and $\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, $\sin\theta=(e^{i\theta}-e^{-i\theta})/2i$.

15b · de Moivre & Roots
WK 2

DE MOIVRE
$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
$(r\,e^{i\theta})^n = r^n e^{in\theta}$

N-TH ROOTS OF $W = R\,e^{i\Phi}$
$z_k = R^{1/n}\exp(i(\varphi+2k\pi)/n)$
$k = 0,1,\ldots,n-1$ (spaced $2\pi/n$ apart)

Roots of unity ($z^n=1$): $e^{2\pi ik/n}$ — n points equally spaced on the unit circle. List all n roots (run k=0…n−1).

Worked. Cube roots of 8: $8=8e^{i0}$, $R^{1/3}=2$, angles $0, 2\pi/3, 4\pi/3 \Rightarrow$ $2, -1+i\sqrt{3}, -1-i\sqrt{3}$.

Polar worked. $z=1+i$: $r=\sqrt{2}$, $\theta=\pi/4 \Rightarrow z=\sqrt{2}\,e^{i\pi/4}$; $z^8 = (\sqrt{2})^8 e^{i2\pi} = 16$.

Argand regions. $|z|=2$ is a circle radius 2; $|z-i|\leq 2$ is a disc centred at i; $\mathrm{Re}(z)>-1$ is a half-plane. Read loci as distances/angles, not coordinates.

de Moivre for identities: expand $(\cos\theta+i\sin\theta)^3$ and match real/imaginary parts $\Rightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Modulus worked. $z=3-4i$: $|z|=\sqrt{(9+16)}=5$; arg in the 4th quadrant, $\theta=\tan^{-1}(-4/3)\approx-53°$. $\bar{z}=3+4i$, $z\bar{z}=25$.

The n roots of unity sum to 0 (for $n\geq2$) and are the vertices of a regular n-gon on the unit circle — a quick sketch checks your answer. One root is always z=1; the rest are its powers ω^k.

16 · Polynomials over $\mathbb{C}$
ROOTS

Complex exponential. $z=a+ib \Rightarrow e^z=e^a(\cos b + i\sin b)$; $|e^z|=e^{\mathrm{Re}\,z}$, $\arg(e^z)=\mathrm{Im}\,z$. Solving $e^z=w$ gives ∞-many solutions (b mod 2π).

Fundamental Theorem of Algebra. Every non-constant p(z) has a root in $\mathbb{C}$; counted with multiplicity, degree n $\Rightarrow$ exactly n roots.

Conjugate root theorem. p with *real* coefficients, α a root $\Rightarrow \bar{\alpha}$ a root. Then $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2$ divides p; divide out to find the rest.

Worked. $z^4+1=0$: roots are the 4th roots of $-1 = e^{i\pi}$, i.e. $e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ — two conjugate pairs, so $z^4+1 = (z^2-\sqrt{2}z+1)(z^2+\sqrt{2}z+1)$ over $\mathbb{R}$.

17 · Vectors in $\mathbb{R}^n$
WK 3–4

$\vec{AB} = \vec{OB} - \vec{OA} = [b_1-a_1,\ldots]$ (head − tail). a+b head-to-tail; ca scales.

LENGTH / UNIT VECTOR
$\|a\| = \sqrt{(a_1^2+\ldots+a_n^2)}$ · $\|ca\|=|c|\,\|a\|$
$\hat{a} = a/\|a\|$

DOT PRODUCT (SCALAR)

U·V & ANGLE
$u\cdot v = u_1v_1+\ldots+u_nv_n$ · $\|u\|=\sqrt{(u\cdot u)}$
$u\cdot v = \|u\|\|v\|\cos\theta \Rightarrow \cos\theta = u\cdot v / (\|u\|\|v\|)$

Orthogonal $\iff u\cdot v=0$. Cauchy–Schwarz $|u\cdot v|\leq\|u\|\|v\|$.

17b · Projection
WK 4

PROJECT V ONTO U
$\mathrm{proj}_u(v) = (u\cdot v / \|u\|^2)\,u = (\hat{u}\cdot v)\,\hat{u}$

Decompose $v = a + b$: $a \parallel u$, $b = v - \mathrm{proj}_u(v) \perp u$.

Distance point P → line (dir u, A on line): $\|\vec{AP} - \mathrm{proj}_u(\vec{AP})\|$.

Trap: divide by $\|u\|^2$, not $\|u\|$; $u\cdot v$ is a scalar, not a vector.

Worked. v=[3,4] onto u=[1,0]: $u\cdot v=3$, $\|u\|^2=1 \Rightarrow$ proj=[3,0]; orthogonal part [0,4]; so dist from (3,4) to the x-axis = 4.

18 · Cross Product
$\mathbb{R}^3$ ONLY · WK 5

U×V
$= [u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1]$
$= \det[\,i\ j\ k\ ;\ u_1\ u_2\ u_3\ ;\ v_1\ v_2\ v_3\,]$

Orthogonal to both u,v (right-hand rule). Anti-commutative $u\times v=-v\times u$; $u\times u=0$; *not* associative. $\|u\times v\| = \|u\|\|v\|\sin\theta$ = parallelogram area.

Worked. u=[1,0,0], v=[0,1,0]: $u\times v = [0,0,1] = k$ (right-hand rule), area = 1. And $v\times u = [0,0,-1] = -(u\times v)$.

Scalar triple $u\cdot(v\times w) = \det[u;v;w]$ = volume of the parallelepiped; zero $\Rightarrow$ the three vectors are coplanar.

Triangle area = ½$\|\vec{AB}\times\vec{AC}\|$. Eg A,B,C with $\vec{AB}$=[1,0,0], $\vec{AC}$=[0,2,0] ⇒ $\|[0,0,2]\|/2$ = 1.

Parallel test: $u \parallel v$ iff $u\times v=0$ (or v=cu). Perpendicular test: $u\perp v$ iff $u\cdot v=0$. Two quick checks for line/plane relations.

Dot worked. u=[1,2,2], v=[2,0,1]: $u\cdot v = 2+0+2 = 4$, $\|u\|=3$, $\|v\|=\sqrt{5} \Rightarrow \cos\theta = 4/(3\sqrt{5}) \approx 0.596 \Rightarrow \theta\approx 53°$.

Orthogonality solve. $[1,k,2]\cdot[3,-1,1]=0 \Rightarrow 3-k+2=0 \Rightarrow$ k=5 makes them perpendicular.

The cross product lives only in $\mathbb{R}^3$; in $\mathbb{R}^2$ or $\mathbb{R}^4$ use dot product, projection and determinants instead. Distance from P to a line = length of the orthogonal component; the sign of $u\cdot v$ shows acute (+) vs obtuse (−) angle, and $u\cdot v=0$ is a right angle.

Cauchy–Schwarz $|u\cdot v|\leq\|u\|\|v\|$ guarantees $\cos\theta\in[-1,1]$, so the angle formula always makes sense.

19 · Lines & Planes
WK 5–6

LINE — VECTOR / PARAMETRIC
$x = p + t\,d$, $t\in\mathbb{R}$ ($d\neq0$ direction)
$x_i = p_i + t\,d_i$ · through A,B: $d = \vec{AB}$

PLANE IN $\mathbb{R}^3$ — NORMAL FORM
$n\cdot(x - p) = 0 \iff ax+by+cz = d$
$n = [a,b,c]$

A plane = point + normal (or point + two non-parallel directions, then $n = d_1\times d_2$). Line in $\mathbb{R}^2$: $n\cdot x = n\cdot p \iff ax+by=c$ with $n\cdot d=0$.

Lines in $\mathbb{R}^3$ have no normal/general form. Relations: coincide / intersect / parallel; in $\mathbb{R}^3$ also **skew**. *Trap:* don't conclude intersection after matching only 2 of 3 coordinates.

20 · Linear Systems
WK 7

Linear eqn $a_1x_1+\ldots+a_nx_n=b$ (no products/powers). **Augmented matrix** [A|b]; homogeneous if all $b_i=0$.

ELEMENTARY ROW OPERATIONS (PRESERVE SOLUTIONS)

1. Swap $R_i\leftrightarrow R_j$
2. Scale $R_i\to cR_i$ ($c\neq0$)
3. $R_i\to R_i+cR_j$

REF: zero rows at bottom, each leading entry right of the one above. **RREF**: + leading 1s, zeros elsewhere in their column (RREF is unique). Gaussian = REF + back-sub; Gauss–Jordan = RREF.

Outcomes: a row [0 … 0 | c], $c\neq0 \Rightarrow$ inconsistent. Else non-leading columns = **free variables** $\Rightarrow$ unique (none) or ∞-many (≥1).

Worked. x+y=3, 2x−y=0: R2−2R1 $\Rightarrow$ −3y=−6 $\Rightarrow$ y=2, x=1. As [A|b]: [1 1|3; 2 −1|0] → [1 1|3; 0 −3|−6] → x=1, y=2.

21 · Matrix Algebra
WK 8

$A=[a_{ij}]$, size m×n. Add (same size) & scale entrywise.

MULTIPLICATION (INNER DIMS MATCH)
$[AB]_{ij} = \Sigma_k\, a_{ik}b_{kj}$ = (row i of A)·(col j of B)
A is p×q, B is q×r ⇒ AB is p×r

$AB \neq BA$ in general; associative; I A = A. AB=O does *not* imply A=O or B=O. System: Ax=b.

Worked. [1 2; 3 4][5; 6] = [1·5+2·6; 3·5+4·6] = [17; 39]. Powers A^k need A square.

Geometry of solutions (3 unknowns): planes meeting in a point (unique) / in a line (∞-many, 1 free var) / not at all (inconsistent).

Plane worked. Through P=(1,0,2) with normal n=[2,−1,3]: 2(x−1)−(y)+3(z−2)=0 $\Rightarrow$ 2x − y + 3z = 8. Two planes are parallel iff their normals are; the angle between planes equals the angle between their normals.

Homogeneous Ax=0 is always consistent (x=0 works); it has a nonzero solution iff there is a free variable iff det A=0 (square A).

Rank. rank(A) = number of leading entries in REF = number of pivots. For an n-variable system: unique solution iff rank = n (no free variables). More unknowns than equations $\Rightarrow$ at least one free variable.

Inconsistent example. x+y=1, x+y=2 reduces to [1 1|1; 0 0|1] — the row [0 0|1] is the contradiction 0=1 $\Rightarrow$ no solution (parallel lines).

EROs never change the solution set, so the REF/RREF you reach is logically the same system — just easier to read off. RREF is unique; REF is not. Set each non-leading variable as a free parameter, then write the leading ones in terms of them.

22 · Transpose & Inverse
WK 9–10

TRANSPOSE
$(A^T)^T=A$ · $(A+B)^T=A^T+B^T$
$(AB)^T = B^TA^T$ (order reverses!)

Inverse: $AB=BA=I \Rightarrow B=A^{-1}$ (unique). $(A^{-1})^{-1}=A$; $(AB)^{-1} = B^{-1}A^{-1}$ (order reverses); $(A^T)^{-1}=(A^{-1})^T$.

2×2 INVERSE
A=[a b; c d] invertible $\iff ad-bc\neq0$
$A^{-1} = 1/(ad-bc)$ · [d -b; -c a]

n×n via row reduction: [A | I] → [R | B]. R=I $\Rightarrow A^{-1}$=B; zero row in R $\Rightarrow$ not invertible. Then $x = A^{-1}b$ solves Ax=b.

2×2 worked. A=[1 2; 3 4], det=4−6=−2 $\Rightarrow A^{-1}$ = (−1/2)[4 −2; −3 1] = [−2 1; 3/2 −1/2].

Check: AA^{-1} should give I — always verify one product.

Symmetric $A=A^T$; the diagonal of A and A^T agree, off-diagonals reflect.

Solve via inverse. A=[1 2; 3 4], b=[5; 6]: $x = A^{-1}b$ = [−2 1; 3/2 −1/2][5;6] = [−4; 9/2].

Order reverses, worked. $(AB)^T = B^TA^T$ and $(AB)^{-1} = B^{-1}A^{-1}$ — write them in reverse to keep the inner dimensions matching; getting the order wrong is a classic exam slip.

Only *square* matrices have inverses/determinants. A non-square or det-0 matrix is singular: Ax=b then has 0 or ∞-many solutions, never a unique one. $(A^{-1})^T = (A^T)^{-1}$, so transpose and inverse commute. The 6 invertibility statements opposite are equivalent — prove one, get all.

22b · Invertibility Equivalences
★ THE BIG LIST

For square A, the following are equivalent:

- A is **invertible**
- Ax=b has a unique solution for every b
- Ax=0 has only x=0
- RREF(A) = I
- det A ≠ 0
- 0 is *not* an eigenvalue of A

Corollary: for square A, BA=I alone forces A^{-1}=B. *Trap:* never write 1/A or "divide" by a matrix.

23 · Determinants
WK 10–11

COFACTOR (LAPLACE) EXPANSION
$\det A = \Sigma_j\,(-1)^{1+j}\,a_{1j}\det(A_{1j})$
2×2: ad − bc

3×3: $a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$. Expand along any row/col (checkerboard signs $(-1)^{i+j}$) — pick one with many zeros.

Triangular shortcut: det = product of diagonal.

3×3 worked. det[2 0 1; 1 3 2; 0 1 1] = 2(3·1−2·1) − 0 + 1(1·1−3·0) = 2(1)+1 = 3 $\Rightarrow$ invertible (expanded along row 1; row 3 has zeros too).

Product rule for det: det(AB)=det A·det B lets you read off $\det(A^k)=(\det A)^k$ and $\det(A^{-1})=1/\det A$ without computing the matrices.

Vandermonde: det of the rows $[1, x_i, \ldots, x_i^{n-1}] = \prod_{i<j}(x_j-x_i)$ — nonzero iff the x_i are distinct.

Cofactor & adjugate: $C_{ij}=(-1)^{i+j}\det(A_{ij})$; A^{-1} = (1/det A)·adj(A), where $\mathrm{adj}(A)=[C_{ij}]^T$ — the general inverse formula behind the 2×2 case. Since $\det(A^T)$=det A, you may expand along a column as easily as a row, and triangular matrices read off in one step (product of the diagonal).

23b · Determinant Properties
EROS & RULES

EFFECT OF ROW OPERATIONS

- Swap two rows: det → −det
- Scale a row by c: det → c·det
- Add a multiple of a row: det **unchanged**

FURTHER PROPERTIES
$\det(cA) = c^n \det A$ (not c det A)
det(AB) = det A · det B
$\det(A^T)$ = det A · $\det(A^{-1})$ = 1/det A

A zero row/column $\Rightarrow$ det=0. A invertible $\iff$ det A ≠ 0. Note det(A+B) ≠ det A + det B. Reduce to triangular, tracking the row-op factors, for big matrices.

24 · Eigenvalues & Eigenvectors
WK 11

For n×n A: scalar λ and **nonzero** v with Av = λv. λ = eigenvalue, v = λ-eigenvector.

CHARACTERISTIC EQUATION
$Av=\lambda v \iff (A-\lambda I)v=0$ has $v\neq0$
$\iff \det(A - \lambda I) = 0$

det(A−λI) = characteristic polynomial (degree n) $\Rightarrow \leq$ n eigenvalues (possibly complex). **Eigenspace** E_λ = solutions of (A−λI)v=0 (row-reduce [A−λI | 0]).

MULTIPLICITIES

Algebraic = multiplicity as a root of char. poly.
Geometric = dim E_λ (free parameters). Always 1 ≤ geom ≤ alg. A invertible $\iff$ 0 not an eigenvalue.

Sum of algebraic multiplicities = n (counting complex roots). If all are simple (n distinct λ), every geom mult is 1 and the matrix diagonalises automatically — no eigenspace check needed.

24b · Eigen · Worked
2×2

A = [2 1; 1 2]. det(A−λI) = $(2-\lambda)^2-1 = \lambda^2-4\lambda+3 = 0 \Rightarrow$ λ = 1, 3.

λ=3: (A−3I)v=0 $\Rightarrow$ [−1 1; 1 −1]v=0 $\Rightarrow$ v=[1,1]. λ=1: v=[1,−1]. Two distinct λ $\Rightarrow$ independent eigenvectors $\Rightarrow$ diagonalisable.

Any nonzero scalar multiple of an eigenvector is also an eigenvector (same λ) — pick the simplest representative (set a free parameter to 1).

Repeated λ can fail: [1 1; 0 1] has λ=1 (alg mult 2) but only a 1-D eigenspace (geom mult 1) $\Rightarrow$ not diagonalisable. Complex λ are allowed: [0 −1; 1 0] has λ=±i.

Trace & det check: $\Sigma\lambda_i$ = trace(A), $\prod\lambda_i$ = det(A) — a fast sanity check on your eigenvalues.

For A=[2 1; 1 2]: trace 4 = 1+3 ✓, det 3 = 1·3 ✓. Triangular matrices give eigenvalues straight off the diagonal.

Multiplicities worked. A=[2 0; 0 2]=2I: char poly $(2-\lambda)^2$, alg mult 2; E_2 is all of $\mathbb{R}^2 \Rightarrow$ geom mult 2 = alg $\Rightarrow$ diagonalisable (already diagonal). Contrast the [1 1;0 1] shear, geom 1 < alg 2.

Complex eigenvalues. A=[0 −1; 1 0] (rotation by 90°): $\det(A-\lambda I)=\lambda^2+1=0 \Rightarrow \lambda=\pm i$. Real matrices can have complex eigenvalues in conjugate pairs — no real eigenvector here.

To verify a candidate v: just compute Av and check it equals λv — faster than re-solving the system.

Eigenvectors for distinct eigenvalues are automatically independent, so they always give an invertible P. Real symmetric matrices are always diagonalisable, with mutually orthogonal eigenvectors and only real eigenvalues — a useful shortcut whenever $A=A^T$ in an exam question.

25 · Diagonalisation
WK 12

A is **diagonalisable** if $A = PDP^{-1}$ (P invertible, D diagonal), i.e. $P^{-1}AP = D$.

THEOREM
$P = [v_1 \ldots v_n]$, $D = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$
iff $Av_i=\lambda_iv_i$ and the v_i independent

Criterion: diagonalisable $\iff$ for *every* eigenvalue, geometric mult = algebraic mult. n distinct eigenvalues $\Rightarrow$ diagonalisable.

PAYOFF — MATRIX POWERS
$A^k = P\,D^k\,P^{-1}$, $D^k = \mathrm{diag}(\lambda_1^k,\ldots,\lambda_n^k)$

Worked. A=[2 1; 1 2]: P=[1 1; 1 −1], D=diag(3,1) $\Rightarrow A^k$ = P diag(3^k,1) P^{-1}, giving each entry = $(3^k\pm1)/2$ — no repeated multiplication needed.

Why it works: in the eigenbasis A just scales each axis by its λ; P changes coordinates into that basis and P^{-1} back. This is the engine behind Markov-chain long-run states and linear recurrences — also how A^{100} becomes a one-line answer instead of 99 multiplications.

25b · Method Recipes
DO THIS IN ORDER

- **Eigen/diag:** det(A−λI)=0 → λ → eigenspaces → if any geom≠alg, *stop* (not diag.) → else assemble P (columns), D (matching order)
- **Solve Ax=b:** row-reduce [A|b] → REF → free vars → back-sub
- **Invert A:** [A|I] → RREF → $[I|A^{-1}]$
- **Plane from 3 pts:** two directions → $n=d_1\times d_2$ → n·(x−p)=0
- **Roots of z^n=w:** polar form → $R^{1/n}$, add 2π/n each root
- **Angle / distance:** $\cos\theta=u\cdot v/(\|u\|\|v\|)$; point→line via the orthogonal component; point→plane via $|n\cdot(P-Q)|/\|n\|$
- **Is it diagonalisable?** distinct λ $\Rightarrow$ yes; else check geom = alg for each repeated λ
- **Sanity:** Σλ = trace, ∏λ = det — check before assembling P, D

26 · High-Yield Traps
DON'T LOSE MARKS

- L'Hôpital only on genuine 0/0 or ∞/∞ — verify first
- Keep the chain-rule inner factor; quotient is f′g−fg′
- Integration: always +C; convert limits in substitution; reduce improper fractions before partial fractions
- FTC variable limit: ×g′(x); sign flips for a lower limit
- Taylor: check $R_n\to0$ / radius before claiming series = f
- Complex: arg from the correct quadrant; list all n roots; conjugate-root needs real coefficients
- Projection uses $\|u\|^2$; cross product is $\mathbb{R}^3$-only & order-sensitive
- $(AB)^{-1}$, $(AB)^T$ **reverse order**; AB≠BA; no matrix division
- $\det(cA)=c^n\det A$; eigenvectors ≠ 0; match P & D column order
- RREF is unique, REF is not; only square matrices have det / inverse

Algebra Formula Belt
SIDE 2

$e^{i\theta}=\cos\theta+i\sin\theta$ · $(re^{i\theta})^n=r^ne^{in\theta}$
n-th roots: $R^{1/n}\exp(i(\varphi+2k\pi)/n)$
$\mathrm{proj}_u(v)=(u\cdot v/\|u\|^2)u$ · $\|u\times v\|=\|u\|\|v\|\sin\theta$
2×2^{-1} = (1/(ad-bc))[d -b; -c a]
$\det(A-\lambda I)=0$ · $A=PDP^{-1} \Rightarrow A^k=PD^kP^{-1}$
Σλ=tr A · ∏λ=det A