0 · Exam Blueprint
READ FIRST

★ **One paper, six disjoint maths worlds.** The final is 60% of the unit AND a hurdle — you must score **≥45/100 on the exam itself** to pass, regardless of coursework.

ITEM	DETAIL
Weight	**60%** · hurdle ≥45%
Coursework	2 assign×10% + 5 quiz×4%
Duration	~3 h 10 min · e-exam
Materials	**closed-book · NO calc**
Formula sheet	provided in paper
Template	~36 questions, fixed slots

~36-slot pattern: ~31 short-answer (1.5–3 marks, answer is an *integer or lowest-terms a/b*, no spaces, no decimals) + 5 long-answer (6 marks: global extrema · Hessian classify · eigen/diagonalise · free-variable system · two-stage Bayes).

This is a REVISION sheet — you cannot bring it in. The exam gives its own formula sheet, so memorise the METHODS, not the formulas: the recipe wins marks, the formula is handed to you.

SIA → *Everything must be hand-computable. Practise Gaussian elimination, char-poly eigenvalues and Bayes by hand until fluent – no calculator on the day.*

1 · Derivatives · Rules
AREA 1 · L4–5

$f'(a)$ = slope of tangent = $\lim_{x\to a} (f(x)-f(a))/(x-a)$. $|x|$ is not differentiable at 0.

F(X)	F'(X)
c (const)	0
x^b	$b\cdot x^{(b-1)}$
a^x	$\ln(a)\cdot a^x$
e^x	e^x
ln x	1/x
$\log_a x$	$1/(\ln(a)\cdot x)$

COMBINING RULES
```
(c·f)'=c·f' · (f±g)'=f'±g'
product: (fg)' = f'g + fg'
chain: (f(g(x)))' = g'(x)·f'(g(x))
```
COMMON CHAINS
```
(e^(cx+d))' = c·e^(cx+d)
(ln(cx+d))' = c/(cx+d)
(a^(cx+d))' = c·ln(a)·a^(cx+d)
```
nth derivative $f^{(n)}$ = differentiate n times ($f''=f^{(2)}$).
Worked tangent slope: for $h(x)=x\cdot e^{(2x)}$, $h'=e^{(2x)}(1+2x) \Rightarrow h'(0)=1$ (product rule, then evaluate).

1b · Function Types
L1–3

Convex: chord lies on/above plot ⇔ $f''\geq 0$. **Concave**: chord below ⇔ $f''\leq 0$. Lines are both.
Transform-plot trick: log-log linear ⇒ power law (slope −a); log-lin linear ⇒ exponential (slope ln a); lin-log ⇒ logarithmic.

LOG & EXPONENTIAL RULES
```
aˣaʸ=a^(x+y) · (aˣ)ʸ=a^(xy) · a⁰=1
logₐ(xy)=logₐx+logₐy · logₐ(xᵇ)=b logₐx
change of base: logₐx = log_b x / log_b a
```
Line through 2 points: $m=(y_2-y_1)/(x_2-x_1)$, then $y=mx+b$ with $b=y_1-mx_1$; zero at $x=-b/m$.
Injective = distinct inputs give distinct outputs; **surjective** = image fills codomain; **bijective** = both ⇔ has an inverse.

2 · Optimisation (1 var)
AREA 1 · L5–6 ★

Stationary points: $f'(s)=0$. Sign of f' gives increase ($f'>0$) / decrease ($f'<0$).

2ND-DERIVATIVE TEST (AT STATIONARY A)
```
f''(a) > 0 ⇒ local min
f''(a) < 0 ⇒ local max
f''(a) = 0 ⇒ inconclusive (x³,x⁴,-x⁴)
```
First-deriv (sign) test: at stationary m, f' goes +→− ⇒ max; −→+ ⇒ min. Inflection = concavity changes.

GLOBAL EXTREMA ON [C,D]
```
candidates = stationary pts + singular pts (f' undef) + endpoints c,d.
Evaluate f at ALL ⇒ largest = max, smallest = min.
```
Shortcut: local min of a convex f is the global min; local max of concave = global max.
Quadratic roots: $x^2+ax+b=0 \Rightarrow x=-a/2\pm\sqrt{(a^2/4-b)}$; real iff $a^2\geq 4b$. $(x-u)(x-v)=x^2-(u+v)x+uv$.

SIA → *The long-answer "global extrema on [a,b]" question forgets the endpoints at your peril – they are candidates too. Always tabulate f at every candidate.*

2b · Worked · Extrema on interval
RENUMBERED

$f'(t)=t^3-5t^2+6t=t(t-2)(t-3)$ on $[-1,3]$. Stationary **t=0,2,3**. $f''=3t^2-10t+6$: $f''(0)=6>0$ (min), $f''(2)=-2<0$ (max), $f''(3)=3>0$ (min). Compare f at {−1,0,2,3} ⇒ pick global max/min by value.
Cubic variant: $f(x)=2x^3-3x^2-12x+4$ on $[-3,3]$ ⇒ stationary $x=-1,2$; global max $(-1,11)$, global min at the endpoint $(-3,-41)$. The endpoint wins — count it.

2c · Worked · Can min surface
APPLIED

Volume $\pi r^2 h = 128\pi \Rightarrow h=128/r^2$. Surface $f(r)=2\pi(r^2+128/r)$; $f'=2\pi(2r-128/r^2)=0 \Rightarrow r^3=64 \Rightarrow r=4$. $f''>0$ (convex) ⇒ global min; $h=128/16=8$.
Page-layout variant: printable $A=(x-4)(y-6)$ with $xy=294 \Rightarrow A=318-6x-1176/x$; $A'=-6+1176/x^2=0 \Rightarrow x=14$, $A''<0$ (concave, so a max), $y=21$.
Convexity-on-interval trap: for $f''=6ax-12$ to be neither convex nor concave on (2,3), require f'' to *change sign* there ⇒ solve for the parameter range, don't just plug one point.
Constrained product, variants: max xy s.t. $2x+3y=60$ ⇒ 150; s.t. $x+3y=60$ ⇒ 300. Same substitute-then-optimise recipe; check it's a max ($f''<0$), not a min.
Singular points (f' undefined, e.g. a corner like |x| at 0) are candidates too — don't only solve $f'=0$. The Extreme Value Theorem guarantees a continuous f on [c,d] attains both extrema.

2d · RSS / least squares
APPLICATION

RESIDUAL SUM OF SQUARES
$$RSS = \sum_i (y_i - f(x_i))^2$$
Fit $f(x)=2x-1$ to (2,1),(3,4),(5,2): preds 3,5,9; residuals −2,−1,−7 ⇒ RSS = 4+1+49 = 54. Squares are differentiable & punish big errors.
Workflow: ① f', solve $f'=0$ + find singular pts; ② classify with f'' (or sign change); ③ compare f at stationary/singular/boundary; use convexity if available.

3 · Integration · FTC
AREA 1 · L7

Antiderivative F: $F'=f$, unique up to +c. $\int f\,dx = F(x)+c$.

F(X)	∫F DX
x^a ($a\neq -1$)	$x^{(a+1)}/(a+1)+c$
x^{-1}	$\ln\|x\|+c$
$e^{(ax)}$	$(1/a)e^{(ax)}+c$

FUNDAMENTAL THEOREM OF CALCULUS
$$\int_a^b f(x)dx = F(b) - F(a)$$
$$G(x)=\int_a^x f(z)dz \Rightarrow G'=f$$
Linearity $\int(f+g)=\int f+\int g$, $\int cf=c\int f$. **Additivity** $\int_a^c=\int_a^b+\int_b^c$ (piecewise).
Rate ⇒ total: if $E'(x)$=rate, total change = ∫ rate dx.

SIA → *A definite integral is signed area – area below the x-axis counts negative. Sketch first if signs are in doubt.*

3b · Worked · Definite + FTC
RENUMBERED

$\int_0^2 (x^3-6x^2)dx = [x^4/4 - 2x^3]_0^2 = 4 - 16 = -12$.
Rate→total (battery): cost falls at rate $(x+1)^{-2}$, start \$5. $D(4)=5+\int_0^4 -(x+1)^{-2}dx = 5 + [(x+1)^{-1}]_0^4 = 5 + (1/5 - 1) =$ **\$4.20** = 21/5. Antiderivative chosen so D(0)=5.
Look-up integrals (used in probability): $\int xe^{-x}dx = -e^{-x}(x+1)+c$; $\int e^{(-x^2/2)}dx$ is not elementary (the normal's normaliser uses erf — hence z-tables). $\int_2(6x^2+6x-4)dx$-type slots are routine power-rule.

3c · Σ / Π Notation
SLOTS 1–2

$\sum_{x=a}^{b} f(x)=f(a)+\ldots+f(b)$; Π is the product. Empty sum = 0, empty product = 1. e.g. $\sum_{k=1}^{4} k^2 = 1+4+9+16 = 30$; $\prod_{k=1}^{4} k = 4! = 24$. $\mathbb{N}=\{0,1,2,\ldots\}$ here (0 is natural). "iff" = if and only if.

4 · Vectors
AREA 2 · L8

$\mathbb{R}^d$ = column d-tuples; add & scale component-wise.

DOT PRODUCT & NORM
```
v·w = v₁w₁+…+v_dw_d
‖v‖ = √(v·v) = √(v₁²+…+v_d²)
orthogonal ⇔ v·w = 0
```
Linear comb. $w=a_1v_1+\ldots+a_nv_n$. Linearly dependent = one v_i is a combo of the rest. Pairwise-orthogonal nonzero vectors are independent.
Worked norm: $\|(3,12,-4)\|=\sqrt{(9+144+16)}=\sqrt{169}=$**13**; $\|(-8,9,-12)\|=\sqrt{289}=$**17**. *Orthogonal solve:* $(2,-1,z)\cdot(3,4,1)=0 \Rightarrow 6-4+z=0 \Rightarrow z=-2$.
Line interval joining u,v = $\{\alpha u+(1-\alpha)v : \alpha\in[0,1]\}$. Geometrically a vector = displacement (direction + length, no fixed position).

4b · Inverse Functions
L2

$g=f^{-1} \Leftrightarrow g(f(x))=x$ and $f(g(y))=y$. $f^{-1}(x)\neq 1/f(x)$. To find f^{-1}: solve $y=f(x)$ for x. **f has an inverse iff bijective** (injective + surjective).
Worked: $f(x)=x^2+4x$ on $[0,\infty)$, find $f^{-1}(32)$: $x^2+4x=32 \Rightarrow x^2+4x-32=0 \Rightarrow (x+8)(x-4)=0 \Rightarrow x=4$ (take the non-negative root).

4c · Constrained Product
SUBSTITUTION

Max xy s.t. $2x+5y=100$: $y=(100-2x)/5 \Rightarrow$ maximise $(1/5)(100x-2x^2)$; deriv $100-4x=0 \Rightarrow x=25$, $y=10 \Rightarrow xy=250$. Substitute the constraint, reduce to one variable, then optimise — also a Lagrange-style multivariable framing.

5 · Matrices & Systems
AREA 2 · L8–10 ★

Mult: A (m×n)·B (n×r) = (m×r); $(AB)_{ij}$ = row i of A · col j of B. $AB\neq BA$ in general. $A(BC)=(AB)C$.

GAUSSIAN ELIMINATION — 3 VALID ROW OPS
```
① swap two rows
② multiply a row by a nonzero k
③ add a multiple of one row to another
```
Row-reduce Ax=b to upper-triangular, then back-substitute. **Apply ops top-to-bottom, NOT simultaneously.**

OUTCOME	SIGNAL
No solution	row 0=3 (contradiction)
Unique	full pivots
∞ many	row 0=0 ⇒ free var t

Free variable ⇒ write solution in **vector form** (point + t·direction). Fewer equations than variables ⇒ usually ∞ many.

6 · Determinant & Inverse
AREA 2 · L10

2×2 DET & INVERSE
```
det[a b; c d] = ad - bc
A⁻¹ = (1/det A)·[d -b; -c a]
```
A invertible ⇔ $\det(A)\neq 0$. $\det(AB)=\det(A)\det(B)$; $\det(I)=1$. Identity I_n: 1s on diagonal, $AI=IA=A$.

SOLVE VIA INVERSE
```
A invertible ⇒ Ax=b has unique x = A⁻¹b
```
Worked: $\det[1,1;-1,1]=1-(-1)=$**2** ⇒ invertible. $\det[1,1;1,1]=0$ ⇒ not invertible (repeated row). Singular-parameter Q: set $ad-bc=0$, solve.
If A is square but **not** invertible, Ax=b has either no solution or infinitely many (never a unique one).

6b · Worked · Ax=b
RENUMBERED

$3x+y=2$, $5x+2y=3$ ⇒ det=1, $A^{-1}=[2,-1;-5,3]$; $x=A^{-1}b=(2\cdot2-1\cdot3, -5\cdot2+3\cdot3)=(1,-1)$. Reuse A^{-1} for many b.

6c · Worked · Gaussian elim
RENUMBERED

Solve $x+2y-z=6$; $-x+y+2z=3$; $x+y-z=8$.
R2+R1, R3−R1: gives $3y+z=9$ and $-y=2 \Rightarrow y=-2$; back-sub $z=9-3y=15$? recheck signs in your own working — the discipline is **clear one column at a time, top-to-bottom**, then back-substitute and verify in the original.

6d · Free-Variable Systems
PARAMETRISE

Underdetermined (fewer equations than unknowns) ⇒ a row collapses to 0=0 ⇒ one **free variable** $t\in\mathbb{R}$.
Express each pivot variable in terms of t, then write the solution set as point + t·direction (a line) — e.g. $(x,y,z,w)=(a,b,c,0)+t(p,q,r,1)$. State "for all $t\in\mathbb{R}$"; two free variables ⇒ a plane.
Two-variable products (Hadamard) exist but are NOT the matrix product used here — always use the row·column rule. $(kA)B=k(AB)$.
Worked 2×2 mult: $[1,2;0,1]\cdot[3;4]=[1\cdot3+2\cdot4; 0\cdot3+1\cdot4]=[11;4]$. Mult is defined only when columns of A = rows of B.
A vector is an m×1 matrix. Identity I_n acts as 1: $AI_n=A$. If $BA=I$ then $AB=I$, so each is the other's inverse. $A(B+D)=AB+AD$ distributes.

7 · Eigenvalues & Eigenvectors
AREA 2 · L11–12 ★

For square A: $Ax=\lambda x$, $x\neq 0$. x = eigenvector, λ = eigenvalue. Zero vector is never an eigenvector; λ=0 can be an eigenvalue.

RECIPE
```
① eigenvalues: det(A - λI) = 0 (char. poly)
② eigenvectors: solve (A - λI)x = 0 (always ∞ many — scalar multiples)
```
If v is an eigenvector so is cv ($c\neq 0$). Char-poly degree = n; roots = the eigenvalues.

DIAGONALISATION
```
A = P D P⁻¹ · P = (v₁|…|vₙ), D = diag(λᵢ)
⇒ Aⁿ = P Dⁿ P⁻¹ (Dⁿ = diag raised to n)
```
Construct when n distinct eigenvalues. **Use:** linear recurrences/Markov processes $a_n=V^n a_0$; long run ruled by the largest eigenvalue (=1 for stochastic); **PageRank** = eigenvector for λ=1.

SIA → *P^{-1} is usually not needed for the long-answer – examiners want eigenvalues, eigenvectors and the assembled P, D. Show det(A−λI)=0 explicitly.*

7b · Worked · Diagonalise 2×2
RENUMBERED

$A=[0,2;-1,3]$: $\det(A-\lambda I)=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2) \Rightarrow \lambda=1,2$.
λ=2: $(A-2I)x=0 \Rightarrow v=[1,1]^T$. λ=1: $v=[2,1]^T$. So **P=[1,2;1,1], D=[2,0;0,1]**, $A=PDP^{-1}$.
Sanity: trace 0+3=3=1+2 ✓; det 0·3−2·(−1)=2=1·2 ✓.

7c · Worked · A^n recurrence
MATRIX POWERS

$c_{n+1}=4c_n+4u_n$, $u_{n+1}=c_n+4u_n \Rightarrow A=[4,4;1,4]$. $\det(A-\lambda I)=\lambda^2-8\lambda+12=(\lambda-2)(\lambda-6) \Rightarrow \lambda=2,6$; eigenvectors $[2,-1]^T,[2,1]^T$. Long-run ratio → dominant eigenvalue **6**.

7d · Diagonal Matrices
WHY DIAG IS EASY

D ($D_{ij}=0$ off-diagonal): D^n raises each diagonal entry to n. That is the whole point of $A=PDP^{-1}$ — push the power onto D where it's trivial, then conjugate back.
Char-poly check: for a 2×2, $\det(A-\lambda I)=\lambda^2-(\text{trace})\lambda+\det$. So $\lambda^2-3\lambda+2$ came from trace 3, det 2. Sum of eigenvalues = trace; product = det — a fast sanity check.

7e · Worked · Eigenvector unknown
SHORT SLOT

Given λ is an eigenvalue and an eigenvector of the form $(x,1,z)^T$, solve $(A-\lambda I)v=0$ row-by-row for x and z (a small linear system). Likewise "find the missing entry of A^{-1}": use $A\cdot A^{-1}=I$ and read off one equation.

7f · Eigen Recipe Recap
STEP LIST

1. Form A−λI; expand **det(A−λI)=0** to the characteristic polynomial
2. Solve for the roots $\lambda_1,\ldots,\lambda_n$ (the eigenvalues)
3. For each λ, row-reduce **(A−λI)x=0**; the free variable gives the eigenvector (pick the neatest scalar multiple)
4. Distinct λ ⇒ assemble $P=(v_1|\ldots|v_n)$, $D=\text{diag}(\lambda_j)$ ⇒ $A=PDP^{-1}$
5. For powers/recurrences quote $\mathbf{A^n=PD^nP^{-1}}$

8 · Multivariable Calculus
AREA 3 · L13–16 ★

Partial derivative f_x: differentiate, treat y as constant; f_y treats x as constant.

GRADIENT
```
∇f = [fₓ ; f_y]
points in direction of steepest increase
∇f ⊥ the level set through the point
```
Level set of value c = {(x,y): f=c}; level curves never cross.
Linear approx: $f(x+\Delta x,y+\Delta y)\approx f+f_x\Delta x+f_y\Delta y$.

STATIONARY POINT
```
∇f = 0 ⇔ fₓ = 0 AND f_y = 0
```
Types: local min, local max, saddle (max one way, min the other). For nice f, mixed partials equal: $f_{xy}=f_{yx}$.

9 · Hessian Test
AREA 3 · L15–16 ★

HESSIAN & DISCRIMINANT
```
H = [fₓₓ fₓy ; f_yx f_yy]
D = det H = fₓₓ·f_yy - (fₓy)²
```

D = DET H	VERDICT
D>0, $f_{xx}>0$	local **min**
D>0, $f_{xx}<0$	local **max**
D<0	saddle
D=0	inconclusive

Convex (2-var): $f_{xx}\geq 0$, $f_{yy}\geq 0$ AND det H≥0 everywhere ⇒ local min = global min.

SIA → *Solve ∇f=0 by factoring, not expanding – e.g. $f_x=x(x-4y)=0$ splits into x=0 or x=4y. Test every stationary point with its own H.*

9b · Worked · Classify all stat. pts
RENUMBERED

$f=2x+x^2+x^2y-2xy^2$: $\nabla f=(2+2x+2xy-2y^2, x^2-4xy)$; $H=[2+2y, 2x-4y; 2x-4y, -4x]$.
$f_y=x(x-4y)=0$. Cases give (0,±1),(−4,−1): det H=−16<0 ⇒ saddles; (−3,−3): det H>0, $f_{xx}>0$ ⇒ local **min**.

9c · Worked · ∇g & det H
SHORT SLOTS

$g=2xy(x^2+y)$: ∇g at (2,1) = (26,24). Stationary of $f=x-\ln(x^2+y^2)$: ∇f=0 ⇒ **(2,0)**. Given $\nabla f=(2xy^2+2x, 2x^2y-3y^2)$ ⇒ det H(1,−1)=**16**.

9d · How the Islands Connect
RECURRING THEMES

- **Saddle points** have no 1-var analogue — they're why the 2-var test needs det H, not just f_{xx}
- **Convexity ⇒ global extremum** in BOTH 1-var ($f''\geq 0$) and 2-var (det H≥0 + $f_{xx}\geq 0$)
- **Matrix multiplication** drives linear systems, A^n eigen-powers, Markov/PageRank AND graph walk-counting (A^k)
- **Gaussian elimination** solves Ax=b AND finds eigenvectors via (A−λI)x=0
- Integration is the engine for continuous probability (pdf normalisation, E, Var on side 2)

Formula Belt
SIDE 1

```
(xᵇ)'=bx^(b-1) · (eˣ)'=eˣ · chain g'·f'(g)
2nd-deriv: f''>0 min, <0 max, =0 ?
∫xᵃ=x^(a+1)/(a+1) · FTC F(b)-F(a)
det₂=ad-bc · Ax=b ⇒ x=A⁻¹b
det(A-λI)=0 · Aⁿ=PDⁿP⁻¹
det H = fₓₓf_yy-fₓy² (<0 saddle)
```

10 · Counting · Basic Rules
AREA 4 · L17

PRODUCT & SUM RULES

AND (sequence) ⇒ multiply: $|S| = \prod k_j$
OR (disjoint cases) ⇒ add: $|S| = \sum |S_j|$
complement: |good| = |S| − |bad|

Keyword test: "and"/"then" ⇒ ×; "or"/disjoint cases ⇒ +. **$n! = 1\cdot 2\cdots n$**; $0! = 1$. Sequences with k_j choices each ⇒ $|S| = \prod k_j$.

11 · Four Selection Types
AREA 4 · L17–18 ★

Draw r from n — ask: order matter? repeats allowed?

	NO REPEAT	WITH REPEAT
Ordered	n!/(n−r)!	n^r
Unordered	C(n,r)	C(n+r−1, r)

BINOMIAL COEFFICIENT

$C(n,r)$ = "n choose r" = $n!/(r!(n-r)!)$

Permutation of all n = n!. Unordered-with-repetition ↔ multisets (stars & bars: r stars, n−1 bars). Ordered-no-repeat = n(n−1)…(n−r+1).

SIA → *Decide the two yes/no questions before writing anything – "order? repeats?" picks the cell. The classic trap is reading "choose a committee" (unordered) as a permutation.*

11b · Worked · Selections
RENUMBERED

Password 10 symbols from 31 = ordered+repeat = 31^{10}. All-distinct = 31·30·…·22 = 31!/21!.

Committee C(12,9)=C(12,3)=220. C(11,6)=462.

Buy 8 loaves, 6 types (unordered+repeat) = C(13,8)=C(13,5)=**1287**.

4-digit all-odd, no repeat: 5·4·3·2 = 120. **3-digit all-even, no repeat, no leading 0:** 4·4·3 = 48.

Alphabetical-order passwords (letters must be in order) ⇒ one per *unordered* selection: no repeats = C(26,10); with repeats = C(35,10).

12 · Binomial Theorem
AREA 4 · L18

EXPANSION & IDENTITIES

$(x+y)^n = \sum_{r=0}^{n} C(n,r)\, x^{n-r} y^r$
row sum: $\sum_r C(n,r) = 2^n$
Pascal: $C(n,r)=C(n-1,r-1)+C(n-1,r)$
symmetry: $C(n,r)=C(n,n-r)$

2^n = total subsets of an n-set. Grid paths across m×k = a binomial coefficient (choose where the up-moves go).

"Express count as a power, find n": $32^7 = (2^5)^7 = 2^{35}$ ⇒ n=35; $27^7 = (3^3)^7 = 3^{21}$ ⇒ n=21. (Rewrite the base as a prime power, then match exponents.) Symmetry C(n,r)=C(n,n−r) keeps the arithmetic small.

12b · Exactly-k & Positions
CHOOSE-THEN-FILL

Count arrangements with **exactly k** of a special kind: choose the positions × fill them × fill the rest, e.g. a 10-symbol string with exactly 3 specials (5 each) and 7 letters (26 each) = $C(10,3)\cdot 5^3 \cdot 26^7$.

"At least one" via complement: count = (total) − (none). e.g. fleets with ≥1 van = C(235,3) − C(171,3) (all fleets minus van-free fleets) — usually far easier than summing cases. Same idea as Pr(Ā)=1−Pr(A).

13 · Inclusion–Exclusion & Pigeonhole
AREA 4 · L18

TWO / THREE SETS

$|A\cup B| = |A|+|B| - |A\cap B|$
$|A\cup B\cup C| = \sum$ singles − $\sum$ pairs + $|A\cap B\cap C|$

None-of-N (general I–E): |U| − Σ singles + Σ pairs − Σ triples + … + $(-1)^N$(all-N). Same structure as Pr(A∪B).

PIGEONHOLE

n items, m boxes, n > m ⇒ some box ≥ 2
generalised ⇒ some box ≥ ⌈n/m⌉

Use: "must two share…?" or "can all be distinct?" — e.g. 10 people, degrees in {0,…,9}: 0 and 9 can't coexist ⇒ two share a friend-count.

Worked I–E: |U|=100, |A|=30, |B|=40, |A∩B|=10 ⇒ satisfying neither = 100−30−40+10 = 40. (Same add-subtract pattern as Pr(A∪B) — the two topics share one identity.)

Three-set "none": $|U|-\Sigma|A_i|+\Sigma|A_i\cap A_j|-|A_1\cap A_2\cap A_3|$. Venn diagrams make the signs obvious. The general rule alternates sign: +singles? no — subtract singles, add pairs, subtract triples…

14 · Probability Foundations
AREA 5 · L19

Sample space S; Pr:S→[0,1], Σ Pr(s)=1. Event A⊆S, Pr(A)=Σ_{x∈A}Pr(x).

CORE RULES

uniform: $\Pr(A) = |A|/|S|$ (pure counting)
complement: $\Pr(\bar A) = 1 - \Pr(A)$
union: $\Pr(A\cup B)=\Pr(A)+\Pr(B)-\Pr(A\cap B)$
independent: $\Pr(A\cap B)=\Pr(A)\cdot\Pr(B)$

Mutually exclusive ⇒ Pr(A∩B)=0 ⇒ Pr(A∪B)=Pr(A)+Pr(B). Independent repeated trials: $\Pr((s_1,s_2))=\Pr(s_1)\Pr(s_2)$.

Worked: Pr(A)=0.3, Pr(B)=0.4, Pr(A∪B)=0.5 ⇒ Pr(A∩B)=0.3+0.4−0.5=0.2; since 0.2≠0.3·0.4=0.12, A and B are not independent.

Each trial yields exactly one outcome; the unit's probability is **discrete** apart from the continuous distributions in L22.

15 · Conditional & Bayes
AREA 5 · L20 ★

CONDITIONAL & MULTIPLICATION

$\Pr(A|B) = \Pr(A\cap B)/\Pr(B)$
⇒ $\Pr(A\cap B) = \Pr(A|B)\cdot\Pr(B)$

TOTAL PROBABILITY & BAYES

$\Pr(B) = \Pr(B|A)\Pr(A) + \Pr(B|\bar A)\Pr(\bar A)$
$\Pr(A|B) = \Pr(B|A)\Pr(A) / \Pr(B)$

Independence ⇔ Pr(A)=Pr(A|B). Use a **tree diagram**; watch the base-rate fallacy (rare-disease tests).

SIA → *The long-answer Bayes is two steps: total probability first (the denominator), then divide. Write both fractions – method marks survive an arithmetic slip.*

15b · Worked · Two-stage Bayes
RENUMBERED

Supplier A: 20% of parts, 20% late. B: 80%, 5% late.

Pr(late)=0.2·0.2+0.8·0.05 = 0.04+0.04 = **0.08**. Pr(A|late)=0.04/0.08 = 1/2.

Commuter variant: walks 10% (late 60%), drives 90% (late 20%). Pr(walk|late)=0.06/(0.06+0.18)=**1/4**; Pr(drove|on-time)=0.72/0.76=18/19.

16 · Random Variables
AREA 5 · L21 ★

RV X: S→ℝ. Distribution = list Pr(X=x). Independent ⇔ Pr(X=x,Y=y)=Pr(X=x)Pr(Y=y) for all x,y.

EXPECTATION & VARIANCE

$E[X] = \sum p_i x_i$ (weighted average)
$\mathrm{Var}[X] = E[(X-\mu)^2] = E[X^2] - \mu^2$
$\sigma = \sqrt{\mathrm{Var}[X]}$

ALGEBRA

$E[X+Y]=E[X]+E[Y]$ (even if dependent!)
$E[kX]=kE[X]$ · $\mathrm{Var}[kX]=k^2\mathrm{Var}[X]$
indep: $E[XY]=E[X]E[Y]$, $\mathrm{Var}[X+Y]=\mathrm{Var}[X]+\mathrm{Var}[Y]$

Linearity-of-expectation (indicator trick): $E[\Sigma X_i]=\Sigma E[X_i]$ works even when the X_i are dependent — e.g. expected #adjacent-odd-pairs in a 4-digit PIN = 3·(1/4) = 3/4.

16b · Worked · E & Var
RENUMBERED

Die X (1–6): E=3.5. Coin game: 2H→+24, 1H→+2, 0H→−8 ⇒ E = 24·¼+2·½−8·¼ = 6+1−2 = 5.

Two indep uniforms: Var(2X+3Y)=4Var(X)+9Var(Y). X∈{3,7,11} probs ¼,¼,½ ⇒ E[X]=8. Var uses $E[X^2]-\mu^2$, not $E[X]^2$.

17 · Distributions
AREA 5 · L22 ★

DIST	E	VAR
Bernoulli(p)	p	p(1−p)
Binomial(n,p)	**np**	**np(1−p)**
Geometric(p)	(1−p)/p	$(1-p)/p^2$
Unif {a..b}	(a+b)/2	$((b-a+1)^2-1)/12$
Unif [a,b]	(a+b)/2	$(b-a)^2/12$
Expon(λ)	1/λ	$1/\lambda^2$
Normal(μ,σ)	μ	σ^2

Binomial $\Pr(X=k)=C(n,k)p^k(1-p)^{n-k}$; **geometric** $\Pr(X=k)=p(1-p)^k$ (k failures first). $Z=(X-\mu)/\sigma$ = standard normal score.

Recognise the model: single success/fail ⇒ Bernoulli; #successes in n trials ⇒ Binomial; #failures before first success ⇒ Geometric; sums/large counts ⇒ tend to Normal (CLT intuition).

17b · Continuous · PDF
NORMALISE!

PDF REQUIREMENTS

f ≥ 0 everywhere AND ∫ f over domain = 1
$\Pr(a\le X\le b)=\int_a^b f\,dx$ · Pr(X=a)=0

$E[X]=\int x f\,dx$; $\mathrm{Var}=\int x^2 f\,dx - (E[X])^2$. **Find k:** h=kx(1−x) on (0,1) ⇒ ∫=k/6=1 ⇒ k=6. Variant $kx^2(1-x)$ ⇒ k=12. Always set ∫f=1 first, then compute moments.

Worked E,Var: $h(x)=(x-1)^2$ on (0,2): ∫h=1 ✓, E[X]=∫xh=**1**, $\mathrm{Var}=\int x^2h-1=1/5$. (Integration is the engine for continuous distributions — Area 1 feeding Area 5.)

Continuous uniform on [a,b]: f=1/(b−a). Exponential(λ) f=λe^(−λx), x≥0 — memoryless, the Poisson waiting time. ≤ and < are interchangeable since Pr(X=a)=0.

17c · Sum of Independent RVs
CONVOLUTION

Distribution of Z=X+Y: sum probabilities over all (x,y) with x+y=z (enumerate the pairs). For a count of successes across n independent Bernoulli(p) trials, the sum is exactly Binomial(n,p).

Law of large numbers (intuition): the average of n iid trials → E[X] as n→∞.

18 · Graphs · Vocabulary
AREA 6 · L23

Graph = vertices + edges (pairs). **Simple** = no loops, no parallel edges. **Walk** = vertex sequence joined by edges; **length** = #steps. **Path** = walk with all-distinct vertices.

Connected = every pair joined by a path. **Cycle** = walk start=end, length ≥3 (simple). **k-regular** = every vertex degree k.

Same graph can be drawn many ways — crossings/curvature don't matter.

19 · Degrees & Handshaking
AREA 6 · L23 ★

HANDSHAKING LEMMA

$\sum \deg(v) = 2\cdot|E|$
⇒ #edges = (Σ deg)/2
⇒ an EVEN number of vertices have odd degree

Degree = #incident edges. Path: endpoints deg 1, internal deg 2; cycle: all deg 2; k-regular: all deg k. Σ deg odd ⇒ no graph; |E| from a valid sequence = Σ/2.

Feasibility tool: if Σ deg is odd, no such graph. No 3-regular graph on 7 vertices (Σ=21 odd).

SIA → *"Can such a graph exist?" ⇒ reach for handshaking: sum the degrees, check it's even, and check it doesn't exceed the max possible (or break tree limits).*

19b · Worked · Degree sequence
RENUMBERED

(1,1,1,1,2,2,4): Σ=12 ⇒ **6 edges**. (1,2,2,2,3,3,3): Σ=16 ⇒ 8 edges. (3,3,3,1,1): Σ=11 odd ⇒ impossible.

3-regular on 10 vertices ⇒ 10·3/2 = **15 edges**.

(3,3,3,3,4): Σ=16 ⇒ 8 edges. (3,2,1,0) on 4 vertices: Σ even but no such graph (deg-3 needs all others as neighbours, contradicting deg-0).

20 · Trees
AREA 6 · L24 ★

Tree = connected + no cycles. Equivalent: any two vertices joined by a *unique* path; deleting any edge disconnects it; adding any edge makes a cycle.

EDGE COUNT

a tree on n vertices has exactly n − 1 edges

So 10 vertices + 8 edges can't be connected. **Spanning tree** = tree inside G with all vertices (delete one edge per cycle). Removable edges keeping connectivity = |E| − (|V|−1). A tree with a Hamilton path *is* a path.

20b · Counting Cycle Graphs
RENUMBERED

How many distinct cycle graphs on {A,B,C,D}? Each vertex has degree 2; fix A's two neighbours (C(3,2)=3 ways) and each choice completes uniquely ⇒ 3 cycles. (Counting meets graphs.)

20c · Graph Drawings
SAME GRAPH

Two drawings are the **same graph** if vertices & edges match — bends, crossings and positions are irrelevant. Deleting any vertex of a cycle leaves a connected path.

Walk vs path: a walk may repeat vertices; a path may not. If any walk A→B exists, a path A→B also exists.

Digraph edges have direction; a multigraph allows loops/parallel edges. A cycle has length ≥3 in a simple graph.

21 · Adjacency Matrix
AREA 6 · L24 ★

A: a_{ij}=1 if v_i adjacent to v_j, else 0. Symmetric for undirected simple graphs. From A: #vertices = dimension; **#edges = (count of 1s)/2**; #components by tracing connectivity.

WALK COUNTING — LINKS TO AREA 2

$(A^k)_{ij}$ = number of walks of length k from v_i to v_j

Compute walk counts by matrix powers (same machinery as eigen/diagonalisation). **Worked read-off:** a 9×9 0/1 matrix with twelve 1s ⇒ 9 vertices, 6 edges; trace components.

$(A^2)_{ii}$ = degree of v_i (number of length-2 closed walks = its neighbours). The diagonal of A is all 0 for a simple graph (no loops).

SIA → *A^k ties graphs back to linear algebra – if you can diagonalise A you can get walk counts for any k. Same toolbox, two topic islands.*

22 · Euler vs Hamilton
AREA 6 · L24

TYPE	VISITS	EXISTS IFF
Euler circuit	every *edge* once, closed	connected & ALL degrees even
Euler trail	every edge once, open	connected & ≤2 odd-deg
Hamilton cycle	every *vertex* once	hard (no easy test)

Euler = edges, Hamilton = vertices. Königsberg's bridges: 4 odd-degree vertices ⇒ no Euler trail.

Hamilton sufficient (Dirac): simple, n≥3, every degree ≥ n/2 ⇒ hamiltonian (not necessary). Hamiltonicity relates to P vs NP.

Mnemonic: Euler walks the *edges* (every bridge once); Hamilton visits the *towns* (every vertex once). Euler has a clean degree test; Hamilton has none easy — that's the whole difference.

A connected graph always has a spanning tree (delete one edge from each cycle until acyclic) — the backbone for counting removable edges.

22b · Worked · Tree feasibility
RENUMBERED

Can a 101-vertex graph with ten vertices of degree 11 be a tree? Σ deg ≥ 11·10 + 91·1 = 201 ⇒ |E| ≥ 201/2 ⇒ > 100 = n−1. No — a tree needs exactly 100 edges.

Cycle from edges: adjacency 1s at (2,3),(2,4),(3,5),(4,5) form the walk 2-3-5-4-2 = a cycle ⇒ not a tree (trees are acyclic).

22c · Distinct-degree trap
PIGEONHOLE

Can 10 people all have a different number of friends? Degrees must lie in {0,…,9}, but degree 0 and degree 9 cannot coexist (the deg-9 person is friends with everyone, contradicting the deg-0 person) ⇒ only 9 usable values for 10 people ⇒ two must share. **No.**

22d · Removable Edges
SPANNING TREE

Edges you can delete while staying connected = **|E| − (|V|−1)** (drop one per independent cycle until only a spanning tree remains). 3-regular on 10 vertices: 15 − 9 = 6 removable.

23 · Method-Trigger Checklist
READ THE VERB

QUESTION SAYS…	REACH FOR
"max/min on [a,b]"	stat pts + endpoints
"max/min/saddle f(x,y)"	∇f=0 then det H
"solve the system"	Gaussian elim
"not invertible"	set det = 0
"A=PDP⁻¹ / Aⁿ"	det(A−λI)=0
"how many ways"	order? repeat? ⇒ cell
"committee / choose"	C(n,r)
"given … find Pr(·|·)"	total prob + Bayes
"E[·] / Var(·) from pdf"	normalise, ∫xf, ∫x²f
"# edges from degrees"	handshaking /2
"walks of length k"	$(A^k)_{ij}$
"can graph exist?"	parity / n−1 edges

23b · Counting → Probability
THE BRIDGE

Uniform probability is pure counting: Pr(A)=|A|/|S|, so every Area-4 selection formula feeds straight into Area 5. Binomial probabilities ARE C(n,k) weighted by $p^k(1-p)^{n-k}$.

Independence test for RVs: one mismatched cell Pr(X=x,Y=y)≠Pr(X=x)Pr(Y=y) disproves independence; check before using Var[X+Y]=Var[X]+Var[Y].

Inclusion–exclusion ↔ Pr(A∪B) are the same identity; **counting ↔ uniform probability** via |A|/|S|; **A^k walk-counting ↔ eigen-powers**. Six islands, one toolbox.

24 · Exam Discipline
DON'T LOSE MARKS

- Short-answer = **integer or lowest-terms a/b, no spaces, no decimals**
- No calculator — keep arithmetic clean, check by substitution
- Long-answer: **show the working**; method marks survive a wrong final number
- Gaussian elim: ops **top-to-bottom**, never simultaneously; verify in the original system
- Global extrema: include the **endpoints**
- Bayes: compute the **denominator (total prob) first**
- Hand-written parts: scan + upload in the 30 min after the e-exam
- Counting: ask **order? repeats?** before picking a formula
- "Is this graph possible?" ⇒ check degree parity and the n−1 edge bound
- Eigen long-answer: show **det(A−λI)=0** and assemble P, D explicitly

SIA → *The formula sheet is given – so spend revision time on recipes, not memorising formulas: be able to run each method end-to-end, by hand, under time.*

Formula Belt
SIDE 2

ordered: $n!/(n-r)!$, n^r · unord: $C(n,r)$, $C(n+r-1,r)$
$(x+y)^n=\Sigma C(n,r)x^{n-r}y^r$ · $\Sigma_r C(n,r)=2^n$
$\Pr(A|B)=\Pr(B|A)\Pr(A)/\Pr(B)$
$\mathrm{Var}[X]=E[X^2]-\mu^2$ · Bin: E=np, Var=np(1−p)
pigeonhole ⌈n/m⌉ · uniform Pr=|A|/|S|
Σ deg=2|E| · tree: n−1 edges · $(A^k)_{ij}$=walks